Distance Learning Initiative

Introduction to Robotics

Robotic Arm Link Accelerations Example

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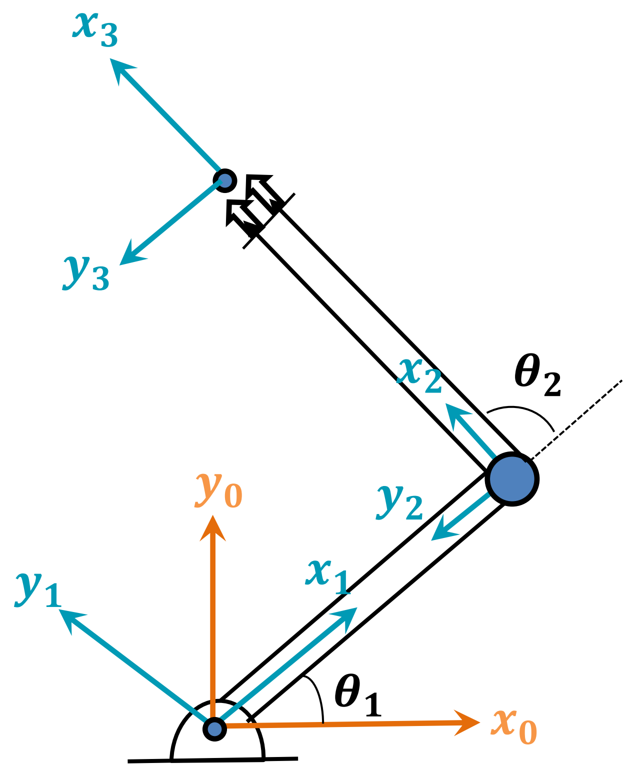
Link Acceleration

Example: For the planar 2 DOF RR robotic arm, calculate the acceleration of each link and that of the end-effector as a function of the joint accelerations?

$$[{}^0_1\mathrm{T}] = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[{}^1_2\mathrm{T}] = \begin{bmatrix} c_2 & -s_2 & 0 & L_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[{}^2_3\mathrm{T}] = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Link Acceleration

$$[{}^0_1\mathrm{T}] = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow [{}^0_1R] = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^1_2\mathrm{T}] = \begin{bmatrix} c_2 & -s_2 & 0 & L_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow [{}^1_2R] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^2_3\mathrm{T}] = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow [{}^2_3R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Link Acceleration

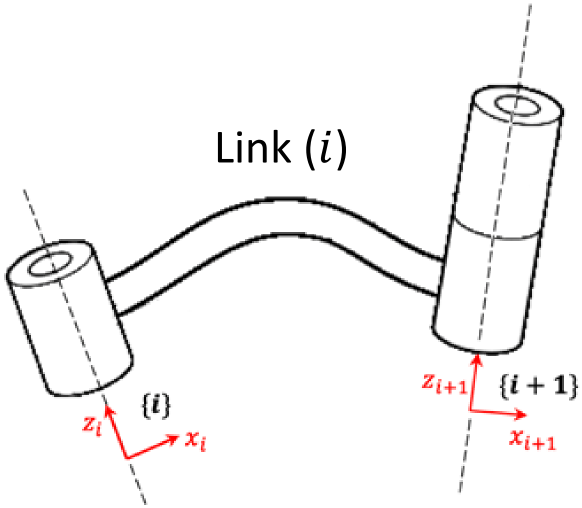
$$[{}^{0}_{1}T] = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow [{}^{0}P_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[{}^{1}_{2}T] = \begin{bmatrix} c_2 & -s_2 & 0 & L_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow [{}^{1}P_2] = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix}$$

$$[{}^{2}_{3}T] = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow [{}^{2}P_3] = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$$

Link Acceleration

If joint $(i+1)$ is a **revolute joint** :



$$\left[{}^{i+1}v_{i+1}\right] = \left[{}^{i+1}_{i}R\right]\left(\left[{}^{i}v_i\right] + \left[S({}^{i}\omega_i)\right]\left[{}^{i}P_{i+1}\right]\right)$$

$$\left[{}^{i+1}\omega_{i+1}\right] = \left[{}^{i+1}_{i}R\right]\left[{}^{i}\omega_i\right] + {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

$$\left[{}^{i+1}a_{i+1}\right] = \left[{}^{i+1}_{i}R\right]\left(\left[{}^{i}a_i\right] + \left[S({}^{i}\alpha_i)\right]\left[{}^{i}P_{i+1}\right] + \left[S({}^{i}\omega_i)\right]\left[S({}^{i}\omega_i)\right]\left[{}^{i}P_{i+1}\right]\right)$$

$$\left[{}^{i+1}\alpha_{i+1}\right] = \left[{}^{i+1}_{i}R\right]\left[{}^{i}\alpha_i\right] + \left[S({}^{i+1}\omega_{i+1})\right] {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} + {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{bmatrix}$$

Link Acceleration

For $i = 0$;

$$[\,{}^1\omega_1] = [{}^1_0R][\,{}^0\omega_0] + \begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix}$$

$$[\,{}^1\omega_1] = \begin{bmatrix}c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix}^T \begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix} = \begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix}$$

$$[\,{}^1\alpha_1] = [{}^1_0R][\,{}^0\alpha_0] + [\,S({}^1\omega_1)] \begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix} + \begin{bmatrix}0\\0\\\ddot\theta_1\end{bmatrix}$$

$$[\,{}^1\alpha_1] = \begin{bmatrix}c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix}^T \begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0 & -\dot\theta_1 & 0\\ \dot\theta_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix} + \begin{bmatrix}0\\0\\\ddot\theta_1\end{bmatrix} = \begin{bmatrix}0\\0\\\ddot\theta_1\end{bmatrix}$$

$$[\,{}^1a_1] = [{}^1_0R]([\,{}^0a_0] + [\,S({}^0\alpha_0)][{}^0P_1] + [\,S({}^0\omega_0)][\,S({}^0\omega_0)][{}^0P_1]\,)$$

$$[\,{}^1a_1] = \begin{bmatrix}c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix}^T \left(\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$$

$$[\,{}^1a_1] = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$[{}^{i+1}\omega_{i+1}] = [{}^{i+1}_{\ \ i}R][{}^i\omega_i] + {}^{i+1}\begin{bmatrix}0\\0\\\dot\theta_{i+1}\end{bmatrix}$$

$$[{}^{i+1}a_{i+1}] = [{}^{i+1}_{\ \ i}R]([\,{}^ia_i] + [\,S({}^i\alpha_i)][{}^iP_{i+1}] + [\,S({}^i\omega_i)][\,S({}^i\omega_i)][{}^iP_{i+1}]\,)$$

$$[{}^{i+1}\alpha_{i+1}] = [{}^{i+1}_{\ \ i}R][\,{}^i\alpha_i] + [\,S({}^{i+1}\omega_{i+1})]\ {}^{i+1}\begin{bmatrix}0\\0\\\dot\theta_{i+1}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot\theta_{i+1}\end{bmatrix}$$

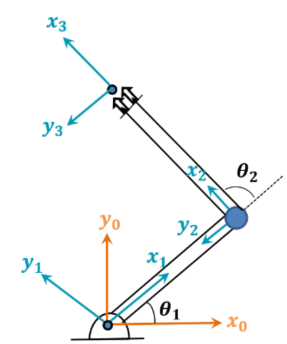
$$[S(a)] = \begin{bmatrix}0 & -a_z & a_y\\ a_z & 0 & -a_x\\ -a_y & a_x & 0\end{bmatrix}$$

$$[{}^0_1R] = \begin{bmatrix}c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[{}^1_2R] = \begin{bmatrix}c_2 & -s_2 & 0\\ s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[{}^2_3R] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[{}^0P_1] = \begin{bmatrix}0\\0\\0\end{bmatrix}\ ;\ [{}^1P_2] = \begin{bmatrix}L_1\\0\\0\end{bmatrix}\ ;\ [{}^2P_3] = \begin{bmatrix}L_2\\0\\0\end{bmatrix}$$



Link Acceleration

$$[{}^1\omega_1] = \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} \quad \text{and} \quad [{}^1\alpha_1] = \begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix} \quad \text{and} \quad [{}^1a_1] = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

For $i = 1$;

$$[{}^2\omega_2] = [{}^2_1R][{}^1\omega_1] + \begin{bmatrix}0\\0\\\dot{\theta}_2\end{bmatrix}$$

$$[{}^2\omega_2] = \begin{bmatrix}c_2 & -s_2 & 0\\ s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}^T \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} + \begin{bmatrix}0\\0\\\dot{\theta}_2\end{bmatrix}$$

$$[{}^2\omega_2] = \begin{bmatrix}c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix} \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} + \begin{bmatrix}0\\0\\\dot{\theta}_2\end{bmatrix} = \begin{bmatrix}0\\0\\\dot{\theta}_1 + \dot{\theta}_2\end{bmatrix}$$

$$[{}^2\alpha_2] = [{}^2_1R][{}^1\alpha_1] + [S({}^2\omega_2)] \begin{bmatrix}0\\0\\\dot{\theta}_2\end{bmatrix} + \begin{bmatrix}0\\0\\\ddot{\theta}_2\end{bmatrix}$$

$$[{}^2\alpha_2] = \begin{bmatrix}c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix} \begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix} + \begin{bmatrix}0 & -(\dot{\theta}_1 + \dot{\theta}_2) & 0\\ (\dot{\theta}_1 + \dot{\theta}_2) & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \begin{bmatrix}0\\0\\\dot{\theta}_2\end{bmatrix} + \begin{bmatrix}0\\0\\\ddot{\theta}_2\end{bmatrix}$$

$$[{}^2\alpha_2] = \begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\ddot{\theta}_2\end{bmatrix} = \begin{bmatrix}0\\0\\\ddot{\theta}_1 + \ddot{\theta}_2\end{bmatrix}$$

$$[{}^{i+1}\omega_{i+1}] = [{}^{i+1}_iR][{}^i\omega_i] + {}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix}$$

$$[{}^{i+1}a_{i+1}]$$
$$= [{}^{i+1}_iR]\left([{}^ia_i] + [S({}^i\alpha_i)][{}^iP_{i+1}] + [S({}^i\omega_i)][S({}^i\omega_i)][{}^iP_{i+1}]\right)$$

$$[{}^{i+1}\alpha_{i+1}] = [{}^{i+1}_iR][{}^i\alpha_i] + [S({}^{i+1}\omega_{i+1})]\ {}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot{\theta}_{i+1}\end{bmatrix}$$

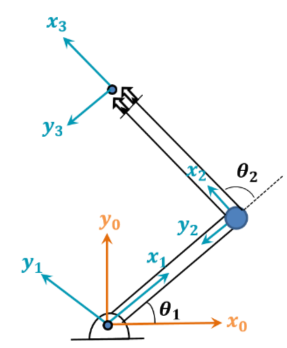
$$[S(a)] = \begin{bmatrix}0 & -a_z & a_y\\ a_z & 0 & -a_x\\ -a_y & a_x & 0\end{bmatrix}$$

$$[{}^0_1R] = \begin{bmatrix}c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[{}^1_2R] = \begin{bmatrix}c_2 & -s_2 & 0\\ s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[{}^2_3R] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[{}^0P_1] = \begin{bmatrix}0\\0\\0\end{bmatrix} \ ; \ [{}^1P_2] = \begin{bmatrix}L_1\\0\\0\end{bmatrix} \ ; \ [{}^2P_3] = \begin{bmatrix}L_2\\0\\0\end{bmatrix}$$



Link Acceleration

$$[{}^1\omega_1] = \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}, [{}^1\alpha_1] = \begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix}, [{}^1a_1] = \begin{bmatrix}0\\0\\0\end{bmatrix}, [{}^2\omega_2] = \begin{bmatrix}0\\0\\\dot{\theta}_1+\dot{\theta}_2\end{bmatrix}, [{}^2\alpha_2] = \begin{bmatrix}0\\0\\\ddot{\theta}_1+\ddot{\theta}_2\end{bmatrix}$$

Continue for $i = 1$;

$$[{}^2a_2] = [{}^2_1R]\left([{}^1a_1] + [S({}^1\alpha_1)][{}^1P_2] + [S({}^1\omega_1)][S({}^1\omega_1)][{}^1P_2]\right)$$

$$[{}^2a_2] = \begin{bmatrix}c_2 & -s_2 & 0\\ s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}^T \left(\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0 & -\ddot{\theta}_1 & 0\\ \ddot{\theta}_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}L_1\\0\\0\end{bmatrix} + \begin{bmatrix}0 & -\dot{\theta}_1 & 0\\ \dot{\theta}_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & -\dot{\theta}_1 & 0\\ \dot{\theta}_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}L_1\\0\\0\end{bmatrix}\right)$$

$$[{}^2a_2] = \begin{bmatrix}c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix}0\\L_1\ddot{\theta}_1\\0\end{bmatrix} + \begin{bmatrix}0 & -\dot{\theta}_1 & 0\\ \dot{\theta}_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\L_1\dot{\theta}_1\\0\end{bmatrix}\right)$$

$$[{}^2a_2] = \begin{bmatrix}c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix}0\\L_1\ddot{\theta}_1\\0\end{bmatrix} + \begin{bmatrix}-L_1(\dot{\theta}_1)^2\\0\\0\end{bmatrix}\right) = \begin{bmatrix}c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-L_1(\dot{\theta}_1)^2\\L_1\ddot{\theta}_1\\0\end{bmatrix}$$

$$[{}^2a_2] = \begin{bmatrix}-c_2L_1(\dot{\theta}_1)^2 + s_2L_1\ddot{\theta}_1\\ s_2L_1(\dot{\theta}_1)^2 + c_2L_1\ddot{\theta}_1\\ 0\end{bmatrix}$$

$$[{}^{i+1}\omega_{i+1}] = [{}^{i+1}_iR][{}^i\omega_i] + {}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix}$$

$$[{}^{i+1}a_{i+1}] = [{}^{i+1}_iR]\left([{}^ia_i] + [S({}^i\alpha_i)][{}^iP_{i+1}] + [S({}^i\omega_i)][S({}^i\omega_i)][{}^iP_{i+1}]\right)$$

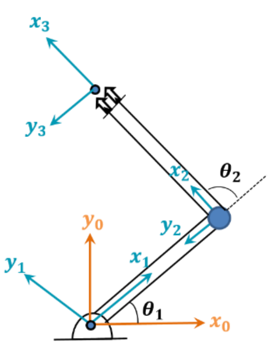
$$[{}^{i+1}\alpha_{i+1}] = [{}^{i+1}_iR][{}^i\alpha_i] + [S({}^{i+1}\omega_{i+1})]\ {}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot{\theta}_{i+1}\end{bmatrix}$$

$$[S(a)] = \begin{bmatrix}0 & -a_z & a_y\\ a_z & 0 & -a_x\\ -a_y & a_x & 0\end{bmatrix}$$

$$[{}^0_1R] = \begin{bmatrix}c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[{}^1_2R] = \begin{bmatrix}c_2 & -s_2 & 0\\ s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[{}^2_3R] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$



$$[{}^0P_1] = \begin{bmatrix}0\\0\\0\end{bmatrix} \; ; \; [{}^1P_2] = \begin{bmatrix}L_1\\0\\0\end{bmatrix} \; ; \; [{}^2P_3] = \begin{bmatrix}L_2\\0\\0\end{bmatrix}$$

Link Acceleration

$$[{}^1\omega_1] = \begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix},\ [{}^1\alpha_1] = \begin{bmatrix}0\\0\\\ddot\theta_1\end{bmatrix},\ [{}^1a_1] = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$[{}^2\omega_2] = \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix},[{}^2\alpha_2] = \begin{bmatrix}0\\0\\\ddot\theta_1+\ddot\theta_2\end{bmatrix},[{}^2a_2] = \begin{bmatrix}-c_2L_1(\dot\theta_1)^2+s_2L_1\ddot\theta_1\\ s_2L_1(\dot\theta_1)^2+c_2L_1\ddot\theta_1\\0\end{bmatrix}$$

For $i=2$:

$$[{}^3\omega_3] = [{}^3_2R][{}^2\omega_2]$$

$$[{}^3\omega_3] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}^T\begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix}$$

$$[{}^3\omega_3] = \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix}$$

$$[{}^3\alpha_3] = [{}^3_2R][{}^2\alpha_2]$$

$$[{}^3\alpha_3] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}^T\begin{bmatrix}0\\0\\\ddot\theta_1+\ddot\theta_2\end{bmatrix}$$

$$[{}^3\alpha_3] = \begin{bmatrix}0\\0\\\ddot\theta_1+\ddot\theta_2\end{bmatrix}$$

$$[{}^{i+1}\omega_{i+1}] = [{}^{i+1}_iR][{}^i\omega_i] + {}^{i+1}\begin{bmatrix}0\\0\\\dot\theta_{i+1}\end{bmatrix}$$

$$[{}^{i+1}a_{i+1}] = [{}^{i+1}_iR]([{}^ia_i] + [S({}^i\alpha_i)][{}^iP_{i+1}] + [S({}^i\omega_i)][S({}^i\omega_i)][{}^iP_{i+1}])$$

$$[{}^{i+1}\alpha_{i+1}] = [{}^{i+1}_iR][{}^i\alpha_i] + [S({}^{i+1}\omega_{i+1})]\ {}^{i+1}\begin{bmatrix}0\\0\\\dot\theta_{i+1}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot\theta_{i+1}\end{bmatrix}$$

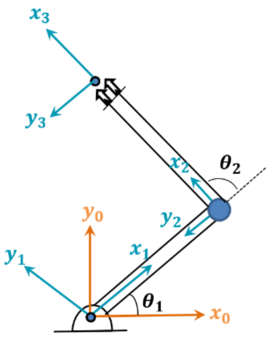
$$[S(a)] = \begin{bmatrix}0&-a_z&a_y\\a_z&0&-a_x\\-a_y&a_x&0\end{bmatrix}$$

$$[{}^0_1R] = \begin{bmatrix}c_1&-s_1&0\\s_1&c_1&0\\0&0&1\end{bmatrix}$$

$$[{}^1_2R] = \begin{bmatrix}c_2&-s_2&0\\s_2&c_2&0\\0&0&1\end{bmatrix}$$

$$[{}^2_3R] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$[{}^0P_1] = \begin{bmatrix}0\\0\\0\end{bmatrix};\ [{}^1P_2] = \begin{bmatrix}L_1\\0\\0\end{bmatrix};\ [{}^2P_3] = \begin{bmatrix}L_2\\0\\0\end{bmatrix}$$



Link Acceleration

$$[{}^1\omega_1] = \begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix},\ [{}^1\alpha_1] = \begin{bmatrix}0\\0\\\ddot\theta_1\end{bmatrix},\ [{}^1a_1] = \begin{bmatrix}0\\0\\0\end{bmatrix},\ [{}^2\omega_2] = \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix},\ [{}^2\alpha_2] = \begin{bmatrix}0\\0\\\ddot\theta_1+\ddot\theta_2\end{bmatrix},$$

$$[{}^2a_2] = \begin{bmatrix}-c_2L_1(\dot\theta_1)^2 + s_2L_1\ddot\theta_1\\ s_2L_1(\dot\theta_1)^2 + c_2L_1\ddot\theta_1\\ 0\end{bmatrix},\ [{}^3\omega_3] = \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix},\ [{}^3\alpha_3] = \begin{bmatrix}0\\0\\\ddot\theta_1+\ddot\theta_2\end{bmatrix}$$

Continue for $i = 2$;

$$[{}^3a_3] = [{}^3_2R]([{}^2a_2] + [S({}^2\alpha_2)][{}^2P_3] + [S({}^2\omega_2)][S({}^2\omega_2)][{}^2P_3])$$

$$[{}^3a_3] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}^T \left(\begin{bmatrix}-c_2L_1(\dot\theta_1)^2 + s_2L_1\ddot\theta_1\\ s_2L_1(\dot\theta_1)^2 + c_2L_1\ddot\theta_1\\ 0\end{bmatrix} + \begin{bmatrix}0 & -(\ddot\theta_1+\ddot\theta_2) & 0\\ (\ddot\theta_1+\ddot\theta_2) & 0 & 0\\ 0&0&0\end{bmatrix}\begin{bmatrix}L_2\\0\\0\end{bmatrix} + \begin{bmatrix}0 & -(\dot\theta_1+\dot\theta_2) & 0\\ (\dot\theta_1+\dot\theta_2) & 0 & 0\\ 0&0&0\end{bmatrix}\begin{bmatrix}0 & -(\dot\theta_1+\dot\theta_2) & 0\\ (\dot\theta_1+\dot\theta_2) & 0 & 0\\ 0&0&0\end{bmatrix}\begin{bmatrix}L_2\\0\\0\end{bmatrix}\right)$$

$$[{}^{i+1}\omega_{i+1}] = [{}^{i+1}_iR][{}^i\omega_i] + {}^{i+1}\begin{bmatrix}0\\0\\\dot\theta_{i+1}\end{bmatrix}$$

$$[{}^{i+1}a_{i+1}] = [{}^{i+1}_iR]([{}^ia_i] + [S({}^i\alpha_i)][{}^iP_{i+1}] + [S({}^i\omega_i)][S({}^i\omega_i)][{}^iP_{i+1}])$$

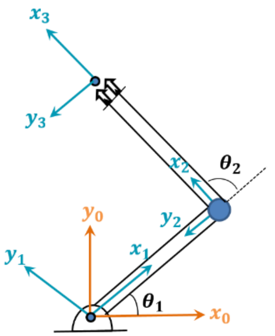
$$[{}^{i+1}\alpha_{i+1}] = [{}^{i+1}_iR][{}^i\alpha_i] + [S({}^{i+1}\omega_{i+1})]\ {}^{i+1}\begin{bmatrix}0\\0\\\dot\theta_{i+1}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot\theta_{i+1}\end{bmatrix}$$

$$[S(a)] = \begin{bmatrix}0 & -a_z & a_y\\ a_z & 0 & -a_x\\ -a_y & a_x & 0\end{bmatrix}$$

$$[{}^0_1R] = \begin{bmatrix}c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0&0&1\end{bmatrix}$$

$$[{}^1_2R] = \begin{bmatrix}c_2 & -s_2 & 0\\ s_2 & c_2 & 0\\ 0&0&1\end{bmatrix}$$

$$[{}^2_3R] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$



$$[{}^0P_1] = \begin{bmatrix}0\\0\\0\end{bmatrix};\ [{}^1P_2] = \begin{bmatrix}L_1\\0\\0\end{bmatrix};\ [{}^2P_3] = \begin{bmatrix}L_2\\0\\0\end{bmatrix}$$

Link Acceleration

$$[\,^{1}\omega_1] = \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}, [\,^{1}\alpha_1] = \begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix}, [\,^{1}a_1] = \begin{bmatrix}0\\0\\0\end{bmatrix}, [\,^{2}\omega_2] = \begin{bmatrix}0\\0\\\dot{\theta}_1+\dot{\theta}_2\end{bmatrix}, [\,^{2}\alpha_2] = \begin{bmatrix}0\\0\\\ddot{\theta}_1+\ddot{\theta}_2\end{bmatrix},$$

$$[\,^{2}a_2] = \begin{bmatrix}-c_2L_1(\dot{\theta}_1)^2 + s_2L_1\ddot{\theta}_1\\ s_2L_1(\dot{\theta}_1)^2 + c_2L_1\ddot{\theta}_1\\ 0\end{bmatrix}, [\,^{3}\omega_3] = \begin{bmatrix}0\\0\\\dot{\theta}_1+\dot{\theta}_2\end{bmatrix}, [\,^{3}\alpha_3] = \begin{bmatrix}0\\0\\\ddot{\theta}_1+\ddot{\theta}_2\end{bmatrix}$$

$$[\,^{3}a_3]$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}^T \left(\begin{bmatrix}-c_2L_1(\dot{\theta}_1)^2 + s_2L_1\ddot{\theta}_1\\ s_2L_1(\dot{\theta}_1)^2 + c_2L_1\ddot{\theta}_1\\ 0\end{bmatrix} + \begin{bmatrix}0 & -(\ddot{\theta}_1+\ddot{\theta}_2) & 0\\ (\ddot{\theta}_1+\ddot{\theta}_2) & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}L_2\\0\\0\end{bmatrix}\right.$$

$$\left. + \begin{bmatrix}0 & -(\dot{\theta}_1+\dot{\theta}_2) & 0\\ (\dot{\theta}_1+\dot{\theta}_2) & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & -(\dot{\theta}_1+\dot{\theta}_2) & 0\\ (\dot{\theta}_1+\dot{\theta}_2) & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}L_2\\0\\0\end{bmatrix}\right)$$

$$[\,^{3}a_3]$$

$$= \begin{bmatrix}-c_2L_1(\dot{\theta}_1)^2 + s_2L_1\ddot{\theta}_1\\ s_2L_1(\dot{\theta}_1)^2 + c_2L_1\ddot{\theta}_1\\ 0\end{bmatrix} + \begin{bmatrix}0\\ L_2(\ddot{\theta}_1+\ddot{\theta}_2)\\ 0\end{bmatrix}$$

$$+ \begin{bmatrix}0 & -(\dot{\theta}_1+\dot{\theta}_2) & 0\\ (\dot{\theta}_1+\dot{\theta}_2) & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\ L_2(\dot{\theta}_1+\dot{\theta}_2)\\ 0\end{bmatrix}$$

$$[\,^{3}a_3] = \begin{bmatrix}-c_2L_1(\dot{\theta}_1)^2 + s_2L_1\ddot{\theta}_1 - L_2(\dot{\theta}_1+\dot{\theta}_2)^2\\ s_2L_1(\dot{\theta}_1)^2 + c_2L_1\ddot{\theta}_1 + L_2(\ddot{\theta}_1+\ddot{\theta}_2)\\ 0\end{bmatrix}$$

$$[\,^{i+1}\omega_{i+1}] = [\,^{i+1}_{\ \ i}R][\,^{i}\omega_i] + {}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix}$$

$$[\,^{i+1}a_{i+1}]$$

$$= [\,^{i+1}_{\ \ i}R]\left([\,^{i}a_i] + [\,S(^{i}\alpha_i)][\,^{i}P_{i+1}] + [\,S(^{i}\omega_i)][\,S(^{i}\omega_i)][\,^{i}P_{i+1}]\right)$$

$$[\,^{i+1}\alpha_{i+1}] = [\,^{i+1}_{\ \ i}R][\,^{i}\alpha_i] + [\,S(^{i+1}\omega_{i+1})]\ {}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot{\theta}_{i+1}\end{bmatrix}$$

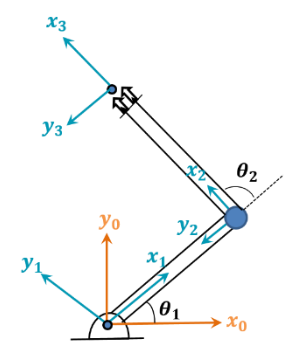
$$[S(a)] = \begin{bmatrix}0 & -a_z & a_y\\ a_z & 0 & -a_x\\ -a_y & a_x & 0\end{bmatrix}$$

$$[\,^{0}_{1}R] = \begin{bmatrix}c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[\,^{1}_{2}R] = \begin{bmatrix}c_2 & -s_2 & 0\\ s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[\,^{2}_{3}R] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[\,^{0}P_1] = \begin{bmatrix}0\\0\\0\end{bmatrix} \ ; \ [\,^{1}P_2] = \begin{bmatrix}L_1\\0\\0\end{bmatrix} \ ; \ [\,^{2}P_3] = \begin{bmatrix}L_2\\0\\0\end{bmatrix}$$



Link Acceleration

$$[\,^{1}\alpha_1] = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}$$

$$[\,^{1}a_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[\,^{2}\alpha_2] = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

$$[\,^{2}a_2] = \begin{bmatrix} -c_2L_1(\dot{\theta}_1)^2 + s_2L_1\ddot{\theta}_1 \\ s_2L_1(\dot{\theta}_1)^2 + c_2L_1\ddot{\theta}_1 \\ 0 \end{bmatrix}$$

$$[\,^{3}\alpha_3] = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

$$[\,^{3}a_3] = \begin{bmatrix} -c_2L_1(\dot{\theta}_1)^2 + s_2L_1\ddot{\theta}_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ s_2L_1(\dot{\theta}_1)^2 + c_2L_1\ddot{\theta}_1 + L_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$$[\,^{i+1}\omega_{i+1}] = [\,^{i+1}_{\;\;i}R][\,^{i}\omega_i] + {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

$$[\,^{i+1}a_{i+1}]$$
$$= [\,^{i+1}_{\;\;i}R]\left([\,^{i}a_i] + [\,S(^{i}\alpha_i)][\,^{i}P_{i+1}] + [\,S(^{i}\omega_i)][\,S(^{i}\omega_i)][\,^{i}P_{i+1}]\right)$$

$$[\,^{i+1}\alpha_{i+1}] = [\,^{i+1}_{\;\;i}R][\,^{i}\alpha_i] + [\,S(^{i+1}\omega_{i+1})]\;{}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} + {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{bmatrix}$$

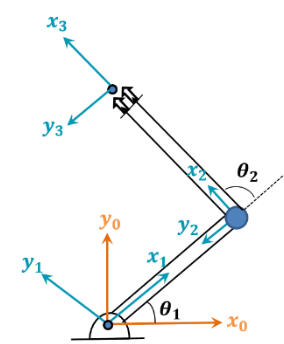
$$[S(a)] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$[\,^{0}_{1}R] = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\,^{1}_{2}R] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\,^{2}_{3}R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\,^{0}P_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \;;\; [\,^{1}P_2] = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \;;\; [\,^{2}P_3] = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$$



Link Acceleration

$$[\,{}^{3}\alpha_3] = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

$$[{}^{0}_{3}R] = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\,{}^{0}\alpha_3] = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

$$\boxed{[\,{}^{0}\alpha_3] = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}}$$

Link Acceleration

$$[\,{}^{3}a_3] = \begin{bmatrix} -c_2L_1(\dot{\theta}_1)^2 + s_2L_1\ddot{\theta}_1 - L_2(\dot{\theta}_1+\dot{\theta}_2)^2 \\ s_2L_1(\dot{\theta}_1)^2 + c_2L_1\ddot{\theta}_1 + L_2(\ddot{\theta}_1+\ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$$[{}^{0}_{3}R] = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\,{}^{0}a_3] = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_2L_1(\dot{\theta}_1)^2 + s_2L_1\ddot{\theta}_1 - L_2(\dot{\theta}_1+\dot{\theta}_2)^2 \\ s_2L_1(\dot{\theta}_1)^2 + c_2L_1\ddot{\theta}_1 + L_2(\ddot{\theta}_1+\ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$$[\,{}^{0}a_3] = \begin{bmatrix} -c_{12}c_2L_1(\dot{\theta}_1)^2 + c_{12}s_2L_1\ddot{\theta}_1 - c_{12}L_2(\dot{\theta}_1+\dot{\theta}_2)^2 - s_{12}s_2L_1(\dot{\theta}_1)^2 - s_{12}c_2L_1\ddot{\theta}_1 - s_{12}L_2(\ddot{\theta}_1+\ddot{\theta}_2) \\ -s_{12}c_2L_1(\dot{\theta}_1)^2 + s_{12}s_2L_1\ddot{\theta}_1 - s_{12}L_2(\dot{\theta}_1+\dot{\theta}_2)^2 + c_{12}s_2L_1(\dot{\theta}_1)^2 + c_{12}c_2L_1\ddot{\theta}_1 + c_{12}L_2(\ddot{\theta}_1+\ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$$[\,{}^{0}a_3] = \begin{bmatrix} -(c_{12}c_2 + s_{12}s_2)L_1(\dot{\theta}_1)^2 - (s_{12}c_2 - c_{12}s_2)L_1\ddot{\theta}_1 - c_{12}L_2(\dot{\theta}_1+\dot{\theta}_2)^2 - s_{12}L_2(\ddot{\theta}_1+\ddot{\theta}_2) \\ -(s_{12}c_2 - c_{12}s_2)L_1(\dot{\theta}_1)^2 + (c_{12}c_2 + s_{12}s_2)L_1\ddot{\theta}_1 - s_{12}L_2(\dot{\theta}_1+\dot{\theta}_2)^2 + c_{12}L_2(\ddot{\theta}_1+\ddot{\theta}_2) \\ 0 \end{bmatrix}$$

Link Acceleration

$$[\,{}^0a_3] = \begin{bmatrix} -(c_{12}c_2 + s_{12}s_2)L_1(\dot\theta_1)^2 - (s_{12}c_2 - c_{12}s_2)L_1\ddot\theta_1 - c_{12}L_2(\dot\theta_1+\dot\theta_2)^2 - s_{12}L_2(\ddot\theta_1+\ddot\theta_2) \\ -(s_{12}c_2 - c_{12}s_2)L_1(\dot\theta_1)^2 + (c_{12}c_2 + s_{12}s_2)L_1\ddot\theta_1 - s_{12}L_2(\dot\theta_1+\dot\theta_2)^2 + c_{12}L_2(\ddot\theta_1+\ddot\theta_2) \\ 0 \end{bmatrix}$$

Trigonometric Identities:
$\cos(a+b) = c_{ab} = c_a c_b - s_a s_b$
$\cos(a-b) = c_a c_b + s_a s_b$
$\sin(a+b) = s_{ab} = s_a c_b + c_a s_b$
$\sin(a-b) = s_a c_b - c_a s_b$
$s_a{}^2 + c_a{}^2 = 1$

$$[\,{}^0a_3] = \begin{bmatrix} -c_1L_1(\dot\theta_1)^2 - s_1L_1\ddot\theta_1 - c_{12}L_2(\dot\theta_1+\dot\theta_2)^2 - s_{12}L_2(\ddot\theta_1+\ddot\theta_2) \\ -s_1L_1(\dot\theta_1)^2 + c_1L_1\ddot\theta_1 - s_{12}L_2(\dot\theta_1+\dot\theta_2)^2 + c_{12}L_2(\ddot\theta_1+\ddot\theta_2) \\ 0 \end{bmatrix}$$

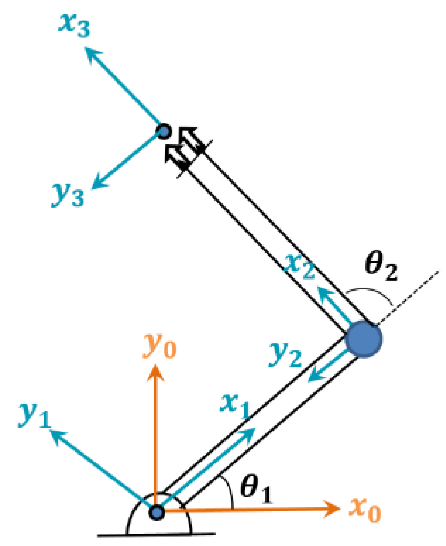
$$[\,{}^0a_3] = \begin{bmatrix} -c_1L_1(\dot\theta_1)^2 - s_1L_1\ddot\theta_1 - c_{12}L_2(\dot\theta_1)^2 - 2c_{12}L_2\dot\theta_1\dot\theta_2 - c_{12}L_2(\dot\theta_2)^2 - s_{12}L_2\ddot\theta_1 - s_{12}L_2\ddot\theta_2 \\ -s_1L_1(\dot\theta_1)^2 + c_1L_1\ddot\theta_1 - s_{12}L_2(\dot\theta_1)^2 - 2s_{12}L_2\dot\theta_1\dot\theta_2 - s_{12}L_2(\dot\theta_2)^2 + c_{12}L_2\ddot\theta_1 + c_{12}L_2\ddot\theta_2 \\ 0 \end{bmatrix}$$

$$[\,{}^0a_3] = \begin{bmatrix} -c_1L_1(\dot\theta_1)^2 - s_1L_1\ddot\theta_1 - c_{12}L_2(\dot\theta_1)^2 - 2c_{12}L_2\dot\theta_1\dot\theta_2 - c_{12}L_2(\dot\theta_2)^2 - s_{12}L_2\ddot\theta_1 - s_{12}L_2\ddot\theta_2 \\ -s_1L_1(\dot\theta_1)^2 + c_1L_1\ddot\theta_1 - s_{12}L_2(\dot\theta_1)^2 - 2s_{12}L_2\dot\theta_1\dot\theta_2 - s_{12}L_2(\dot\theta_2)^2 + c_{12}L_2\ddot\theta_1 + c_{12}L_2\ddot\theta_2 \\ 0 \end{bmatrix}$$

$$[\,{}^0a_3] = \begin{bmatrix} -(c_1L_1 + c_{12}L_2) & -c_{12}L_2 \\ -(s_1L_1 + s_{12}L_2) & -s_{12}L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\dot\theta_1)^2 \\ (\dot\theta_2)^2 \end{bmatrix} + \begin{bmatrix} -(s_1L_1 + s_{12}L_2) & -s_{12}L_2 \\ (c_1L_1 + c_{12}L_2) & c_{12}L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \begin{bmatrix} -2c_{12}L_2 \\ -2s_{12}L_2 \\ 0 \end{bmatrix} [\dot\theta_1\dot\theta_2]$$

Link Acceleration

Summary:



$$[\,{}^{0}\alpha_3] = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

$$[\,{}^{0}a_3] = \begin{bmatrix} -(c_1L_1 + c_{12}L_2) & -c_{12}L_2 \\ -(s_1L_1 + s_{12}L_2) & -s_{12}L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\dot{\theta}_1)^2 \\ (\dot{\theta}_2)^2 \end{bmatrix} + \begin{bmatrix} -(s_1L_1 + s_{12}L_2) & -s_{12}L_2 \\ (c_1L_1 + c_{12}L_2) & c_{12}L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -2c_{12}L_2 \\ -2s_{12}L_2 \\ 0 \end{bmatrix} [\dot{\theta}_1\dot{\theta}_2]$$

Link Acceleration

Another approach: (You can use this to double check!)
From the Jacobian Lecture:

$$ {}^{0}\begin{bmatrix}\dot{x}_3\\ \dot{y}_3\end{bmatrix} = \begin{bmatrix}-(L_1s_1 + L_2s_{12}) & -L_2s_{12}\\ (L_1c_1 + L_2c_{12}) & L_2c_{12}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} $$

$$ {}^{0}\begin{bmatrix}\ddot{x}_3\\ \ddot{y}_3\end{bmatrix} = \begin{bmatrix}-(L_1c_1\dot{\theta}_1 + L_2c_{12}(\dot{\theta}_1 + \dot{\theta}_2)) & -L_2c_{12}(\dot{\theta}_1 + \dot{\theta}_2)\\ -(L_1s_1\dot{\theta}_1 + L_2s_{12}(\dot{\theta}_1 + \dot{\theta}_2)) & -L_2s_{12}(\dot{\theta}_1 + \dot{\theta}_2)\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} + \begin{bmatrix}-(L_1s_1 + L_2s_{12}) & -L_2s_{12}\\ (L_1c_1 + L_2c_{12}) & L_2c_{12}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} $$

$$ {}^{0}\begin{bmatrix}\ddot{x}_3\\ \ddot{y}_3\end{bmatrix} = \begin{bmatrix}-(L_1c_1(\dot{\theta}_1)^2 + L_2c_{12}\left((\dot{\theta}_1)^2 + \dot{\theta}_1\dot{\theta}_2\right)) - L_2c_{12}\left(\dot{\theta}_1\dot{\theta}_2 + (\dot{\theta}_2)^2\right)\\ -(L_1s_1(\dot{\theta}_1)^2 + L_2s_{12}\left((\dot{\theta}_1)^2 + \dot{\theta}_1\dot{\theta}_2\right)) - L_2s_{12}\left(\dot{\theta}_1\dot{\theta}_2 + (\dot{\theta}_2)^2\right)\end{bmatrix} + \begin{bmatrix}-(L_1s_1 + L_2s_{12}) & -L_2s_{12}\\ (L_1c_1 + L_2c_{12}) & L_2c_{12}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} $$

Link Acceleration

$$
{}^{0}\begin{bmatrix}\ddot{x}_3\\ \ddot{y}_3\end{bmatrix} = \begin{bmatrix} -(L_1c_1(\dot{\theta}_1)^2 + L_2c_{12}\left((\dot{\theta}_1)^2 + \dot{\theta}_1\dot{\theta}_2\right)) - L_2c_{12}\left(\dot{\theta}_1\dot{\theta}_2 + (\dot{\theta}_2)^2\right) \\ -(L_1s_1(\dot{\theta}_1)^2 + L_2s_{12}\left((\dot{\theta}_1)^2 + \dot{\theta}_1\dot{\theta}_2\right)) - L_2s_{12}\left(\dot{\theta}_1\dot{\theta}_2 + (\dot{\theta}_2)^2\right) \end{bmatrix}
$$

$$
+ \begin{bmatrix} -(L_1s_1 + L_2s_{12}) & -L_2s_{12} \\ (L_1c_1 + L_2c_{12}) & L_2c_{12} \end{bmatrix} \begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}
$$

$$
{}^{0}\begin{bmatrix}\ddot{x}_3\\ \ddot{y}_3\end{bmatrix} = \begin{bmatrix} -(L_1c_1 + L_2c_{12})(\dot{\theta}_1)^2 - 2L_2c_{12}(\dot{\theta}_1\dot{\theta}_2) - L_2c_{12}(\dot{\theta}_2)^2 \\ -(L_1s_1 + L_2s_{12})(\dot{\theta}_1)^2 - 2L_2s_{12}(\dot{\theta}_1\dot{\theta}_2) - L_2s_{12}(\dot{\theta}_2)^2 \end{bmatrix}
$$

$$
+ \begin{bmatrix} -(L_1s_1 + L_2s_{12}) & -L_2s_{12} \\ (L_1c_1 + L_2c_{12}) & L_2c_{12} \end{bmatrix} \begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}
$$

$$
{}^{0}\begin{bmatrix}\ddot{x}_3\\ \ddot{y}_3\end{bmatrix} = \begin{bmatrix} -(L_1c_1 + L_2c_{12}) & -L_2c_{12} \\ -(L_1s_1 + L_2s_{12}) & -L_2s_{12} \end{bmatrix} \begin{bmatrix}(\dot{\theta}_1)^2\\ (\dot{\theta}_2)^2\end{bmatrix} + \begin{bmatrix}-2L_2c_{12}\\ -2L_2s_{12}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\dot{\theta}_2\end{bmatrix}
$$

$$
+ \begin{bmatrix} -(L_1s_1 + L_2s_{12}) & -L_2s_{12} \\ (L_1c_1 + L_2c_{12}) & L_2c_{12} \end{bmatrix} \begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}
$$

Link Acceleration

Example: For the planar 2 DOF RR robotic arm, calculate the acceleration of each link and that of the end-effector as a function of the joint accelerations? Use vector notations.

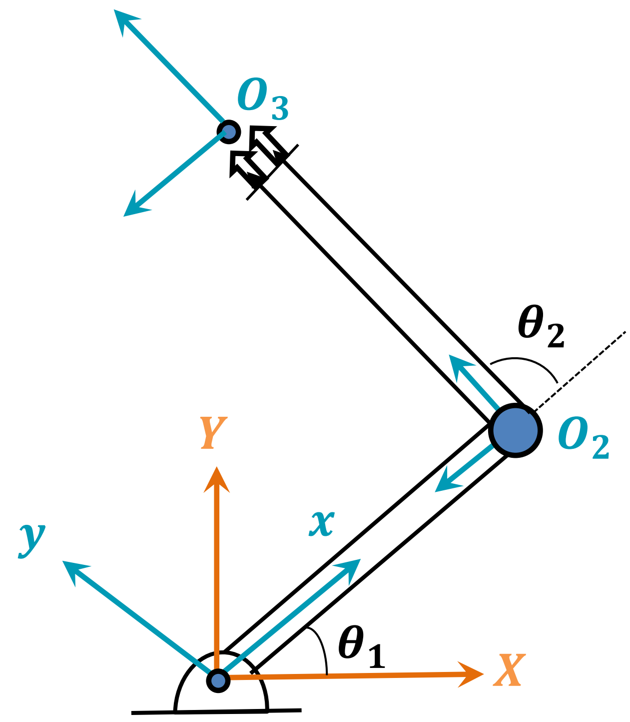
$$\vec{a}_2 = \vec{a}_1 + \vec{\alpha}_1 \times \vec{r}_{2/1} + \vec{\omega}_1 \times \left(\vec{\omega}_1 \times \vec{r}_{2/1}\right) + 2\vec{\omega}_1 \times \left(\vec{v}_{2/1}\right)_{xyz} + \left(\vec{a}_{2/1}\right)_{xyz}$$

$$\vec{a}_2 = \vec{\alpha}_1 \times \vec{r}_{2/1} - (\omega_1)^2\, \vec{r}_{2/1}$$

$$\vec{a}_2 = \alpha_1 \hat{k} \times (L_1 c_1 \hat{\imath} + L_1 s_1 \hat{\jmath}) - (\omega_1)^2 (L_1 c_1 \hat{\imath} + L_1 s_1 \hat{\jmath})$$

$$\vec{a}_2 = \alpha_1 L_1 c_1 \hat{\jmath} - \alpha_1 L_1 s_1 \hat{\imath} - (\omega_1)^2 L_1 c_1 \hat{\imath} - (\omega_1)^2 L_1 s_1 \hat{\jmath}$$

$$\vec{a}_2 = -(\alpha_1 L_1 s_1 + (\omega_1)^2 L_1 c_1)\hat{\imath} + (\alpha_1 L_1 c_1 - (\omega_1)^2 L_1 s_1)\hat{\jmath}$$



Link Acceleration

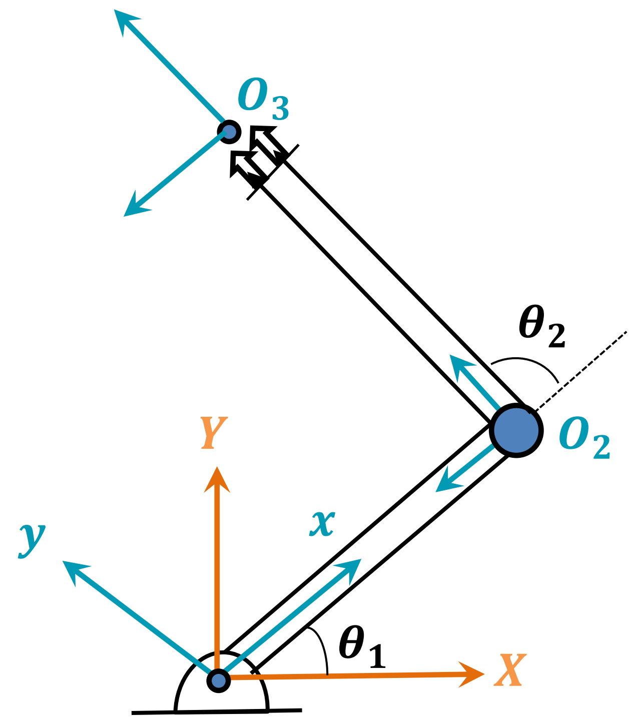
$$\vec{a}_2 = -(\alpha_1 L_1 s_1 + (\omega_1)^2 L_1 c_1)\hat{\imath} + (\alpha_1 L_1 c_1 - (\omega_1)^2 L_1 s_1)\hat{\jmath}$$

$$\vec{a}_3 = \vec{a}_2 + \vec{\alpha}_2 \times \vec{r}_{3/2} - (\omega_2)^2\, \vec{r}_{3/2}$$

$$\vec{a}_3 = -(\alpha_1 L_1 s_1 + (\omega_1)^2 L_1 c_1)\hat{\imath} + (\alpha_1 L_1 c_1 - (\omega_1)^2 L_1 s_1)\hat{\jmath} + \alpha_2 \hat{k} \times (L_2 c_{12}\hat{\imath} + L_2 s_{12}\hat{\jmath}) - (\omega_2)^2 (L_2 c_{12}\hat{\imath} + L_2 s_{12}\hat{\jmath})$$

$$\vec{a}_3 = -(\alpha_1 L_1 s_1 + (\omega_1)^2 L_1 c_1)\hat{\imath} + (\alpha_1 L_1 c_1 - (\omega_1)^2 L_1 s_1)\hat{\jmath} + \alpha_2 L_2 c_{12}\hat{\jmath} - \alpha_2 L_2 s_{12}\hat{\imath} - (\omega_2)^2 L_2 c_{12}\hat{\imath} - (\omega_2)^2 L_2 s_{12}\hat{\jmath}$$

$$\boxed{\vec{a}_3 = -(\alpha_1 L_1 s_1 + \alpha_2 L_2 s_{12} + (\omega_1)^2 L_1 c_1 + (\omega_2)^2 L_2 c_{12})\hat{\imath} + (\alpha_1 L_1 c_1 + \alpha_2 L_2 c_{12} - (\omega_1)^2 L_1 s_1 - (\omega_2)^2 L_2 s_{12})\hat{\jmath}}$$



Link Acceleration

$$\vec{a}_3 = -(\alpha_1 L_1 s_1 + \alpha_2 L_2 s_{12} + (\omega_1)^2 L_1 c_1 + (\omega_2)^2 L_2 c_{12})\hat{\imath} + (\alpha_1 L_1 c_1 + \alpha_2 L_2 c_{12} - (\omega_1)^2 L_1 s_1 - (\omega_2)^2 L_2 s_{12})\hat{\jmath}$$

Substitute for : $\omega_1 = \dot{\theta}_1$, $\omega_2 = \dot{\theta}_1 + \dot{\theta}_2$, $\alpha_1 = \ddot{\theta}_1$, $\alpha_2 = \ddot{\theta}_1 + \ddot{\theta}_2$

$$\vec{a}_3 = -\left(\ddot{\theta}_1 L_1 s_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) L_2 s_{12} + \left(\dot{\theta}_1\right)^2 L_1 c_1 + \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 L_2 c_{12}\right)\hat{\imath} + \left(\ddot{\theta}_1 L_1 c_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) L_2 c_{12} - \left(\dot{\theta}_1\right)^2 L_1 s_1 - \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 L_2 s_{12}\right)\hat{\jmath}$$

$$\vec{a}_3 = -\left(\ddot{\theta}_1 L_1 s_1 + \ddot{\theta}_1 L_2 s_{12} + \ddot{\theta}_2 L_2 s_{12} + \left(\dot{\theta}_1\right)^2 L_1 c_1 + \left(\dot{\theta}_1\right)^2 L_2 c_{12} + 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 c_{12} + \left(\dot{\theta}_2\right)^2 L_2 c_{12}\right)\hat{\imath} + \left(\ddot{\theta}_1 L_1 c_1 + \ddot{\theta}_1 L_2 c_{12} + \ddot{\theta}_2 L_2 c_{12} - \left(\dot{\theta}_1\right)^2 L_1 s_1 - \left(\dot{\theta}_1\right)^2 L_2 s_{12} - 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 s_{12} - \left(\dot{\theta}_2\right)^2 L_2 s_{12}\right)\hat{\jmath}$$

$$\vec{a}_3 = -\left(\ddot{\theta}_1 (L_1 s_1 + L_2 s_{12}) + \ddot{\theta}_2 L_2 s_{12} + \left(\dot{\theta}_1\right)^2 (L_1 c_1 + L_2 c_{12}) + 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 c_{12} + \left(\dot{\theta}_2\right)^2 L_2 c_{12}\right)\hat{\imath} + \left(\ddot{\theta}_1 (L_1 c_1 + L_2 c_{12}) + \ddot{\theta}_2 L_2 c_{12} - \left(\dot{\theta}_1\right)^2 (L_1 s_1 + L_2 s_{12}) - 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 s_{12} - \left(\dot{\theta}_2\right)^2 L_2 s_{12}\right)\hat{\jmath}$$

Link Acceleration

$\vec{a}_3$
$$= -\left(\ddot{\theta}_1(L_1 s_1 + L_2 s_{12}) + \ddot{\theta}_2 L_2 s_{12} + \left(\dot{\theta}_1\right)^2 (L_1 c_1 + L_2 c_{12}) + 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 c_{12} + \left(\dot{\theta}_2\right)^2 L_2 c_{12}\right)\hat{\imath}$$
$$+ \left(\ddot{\theta}_1(L_1 c_1 + L_2 c_{12}) + \ddot{\theta}_2 L_2 c_{12} - \left(\dot{\theta}_1\right)^2 (L_1 s_1 + L_2 s_{12}) - 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 s_{12} - \left(\dot{\theta}_2\right)^2 L_2 s_{12}\right)\hat{\jmath}$$

$(a_3)_x \hat{\imath} + (a_3)_y \hat{\jmath}$
$$= -\left(\ddot{\theta}_1(L_1 s_1 + L_2 s_{12}) + \ddot{\theta}_2 L_2 s_{12} + \left(\dot{\theta}_1\right)^2 (L_1 c_1 + L_2 c_{12}) + 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 c_{12} + \left(\dot{\theta}_2\right)^2 L_2 c_{12}\right)\hat{\imath}$$
$$+ \left(\ddot{\theta}_1(L_1 c_1 + L_2 c_{12}) + \ddot{\theta}_2 L_2 c_{12} - \left(\dot{\theta}_1\right)^2 (L_1 s_1 + L_2 s_{12}) - 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 s_{12} - \left(\dot{\theta}_2\right)^2 L_2 s_{12}\right)\hat{\jmath}$$

$$(a_3)_x = -\left(\ddot{\theta}_1(L_1 s_1 + L_2 s_{12}) + \ddot{\theta}_2 L_2 s_{12} + \left(\dot{\theta}_1\right)^2 (L_1 c_1 + L_2 c_{12}) + 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 c_{12} + \left(\dot{\theta}_2\right)^2 L_2 c_{12}\right)$$
$$(a_3)_y = \ddot{\theta}_1(L_1 c_1 + L_2 c_{12}) + \ddot{\theta}_2 L_2 c_{12} - \left(\dot{\theta}_1\right)^2 (L_1 s_1 + L_2 s_{12}) - 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 s_{12} - \left(\dot{\theta}_2\right)^2 L_2 s_{12}$$

In matrix form:

$$\begin{bmatrix} (a_3)_x \\ (a_3)_y \end{bmatrix} = \begin{bmatrix} -\ddot{\theta}_1(L_1 s_1 + L_2 s_{12}) - \ddot{\theta}_2 L_2 s_{12} - \left(\dot{\theta}_1\right)^2 (L_1 c_1 + L_2 c_{12}) - 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 c_{12} - \left(\dot{\theta}_2\right)^2 L_2 c_{12} \\ \ddot{\theta}_1(L_1 c_1 + L_2 c_{12}) + \ddot{\theta}_2 L_2 c_{12} - \left(\dot{\theta}_1\right)^2 (L_1 s_1 + L_2 s_{12}) - 2\left(\dot{\theta}_1 \dot{\theta}_2\right) L_2 s_{12} - \left(\dot{\theta}_2\right)^2 L_2 s_{12} \end{bmatrix}$$

Link Acceleration

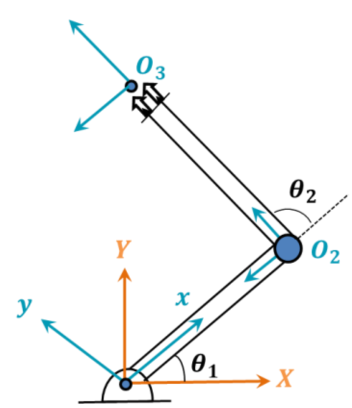
$$\begin{bmatrix}(a_3)_x\\(a_3)_y\end{bmatrix}$$

$$= \begin{bmatrix}-\ddot{\theta}_1(L_1s_1+L_2s_{12})-\ddot{\theta}_2L_2s_{12}-\left(\dot{\theta}_1\right)^2(L_1c_1+L_2c_{12})-2\left(\dot{\theta}_1\dot{\theta}_2\right)L_2c_{12}-\left(\dot{\theta}_2\right)^2L_2c_{12}\\ \ddot{\theta}_1(L_1c_1+L_2c_{12})+\ddot{\theta}_2L_2c_{12}-\left(\dot{\theta}_1\right)^2(L_1s_1+L_2s_{12})-2\left(\dot{\theta}_1\dot{\theta}_2\right)L_2s_{12}-\left(\dot{\theta}_2\right)^2L_2s_{12}\end{bmatrix}$$

$$\begin{bmatrix}(a_3)_x\\(a_3)_y\end{bmatrix}$$

$$= \begin{bmatrix}-(L_1s_1+L_2s_{12}) & -L_2s_{12}\\ (L_1c_1+L_2c_{12}) & L_2c_{12}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}+\begin{bmatrix}-(L_1c_1+L_2c_{12}) & -L_2c_{12}\\ -(L_1s_1+L_2s_{12}) & -L_2s_{12}\end{bmatrix}\begin{bmatrix}\left(\dot{\theta}_1\right)^2\\ \left(\dot{\theta}_2\right)^2\end{bmatrix}$$

$$+\begin{bmatrix}-2L_2c_{12}\\ -2L_2s_{12}\end{bmatrix}\left[\dot{\theta}_1\dot{\theta}_2\right]$$



Link Acceleration

For more details on this subject, please see:

- Introduction to Robotics: Mechanics and Control, by John J. Craig, 3rd Edition, Addison-Wesley Publishing Company, 2003.