Distance Learning Initiative

Introduction to Robotics

Robotic Arm Link Accelerations

Dr. Firas Jarrar

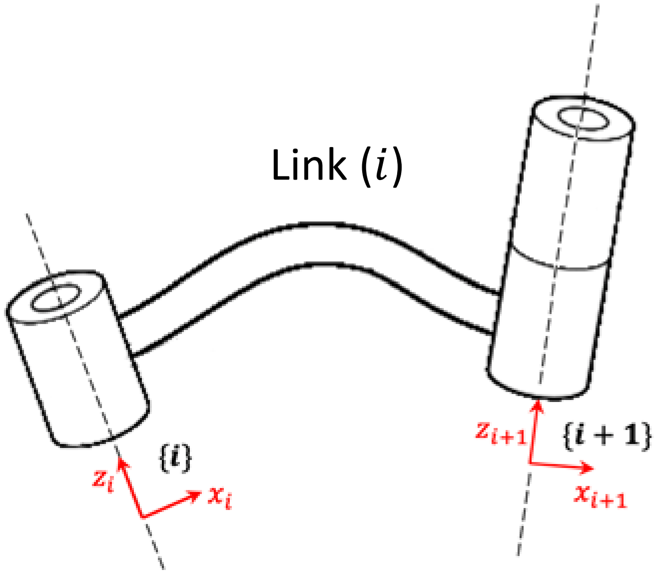
March 17th of 2020

Link Acceleration

Question: How to find the acceleration of a certain link in a robotic arm?

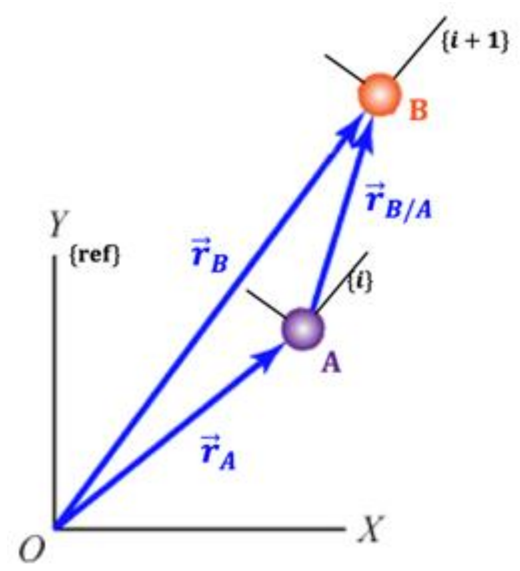
- The linear acceleration (a) of the origin of the frame attached to that link, and

- The rotational acceleration of the link (α)



Link Acceleration

Assume that you have two particles (*A*) and (*B*):



$$[{}^{\text{ref}}r_B] = [{}^{\text{ref}}r_A] + [{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}]$$

Take the derivative of both sides with respect to (t)

$$[{}^{\text{ref}}v_B] = [{}^{\text{ref}}v_A] + [S({}^{\text{ref}}\omega_i)][{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}] + [{}^{\text{ref}}_{i}R][{}^{i}v_{B/A}]$$

Note:

$$\frac{d[R]}{dt} = [S(\omega)][R]$$

$[S(\omega)]$ → A skew-symmetric matrix which takes the following form:

$$[S(\omega)] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Link Acceleration

$$[{}^{\text{ref}}v_B] = [{}^{\text{ref}}v_A] + [S({}^{\text{ref}}\omega_i)]([{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}]) + [{}^{\text{ref}}_{i}R][{}^{i}v_{B/A}]$$

Take the derivative w.r.t. time:

$$[{}^{\text{ref}}a_B]$$

$$= [{}^{\text{ref}}a_A] + [S({}^{\text{ref}}\alpha_i)]([{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}]) + [S({}^{\text{ref}}\omega_i)]([S({}^{\text{ref}}\omega_i)][{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}] + [{}^{\text{ref}}_{i}R][{}^{i}v_{B/A}])$$

$$+ [S({}^{\text{ref}}\omega_i)][{}^{\text{ref}}_{i}R][{}^{i}v_{B/A}] + [{}^{\text{ref}}_{i}R][{}^{i}a_{B/A}]$$

$$[{}^{\text{ref}}a_B]$$

$$= [{}^{\text{ref}}a_A] + [S({}^{\text{ref}}\alpha_i)][{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}] + [S({}^{\text{ref}}\omega_i)][S({}^{\text{ref}}\omega_i)][{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}] + \underline{[S({}^{\text{ref}}\omega_i)][{}^{\text{ref}}_{i}R][{}^{i}v_{B/A}]}$$

$$+ \underline{[S({}^{\text{ref}}\omega_i)][{}^{\text{ref}}_{i}R][{}^{i}v_{B/A}]} + [{}^{\text{ref}}_{i}R][{}^{i}a_{B/A}]$$

$$[{}^{\text{ref}}a_B]$$

$$= [{}^{\text{ref}}a_A] + [S({}^{\text{ref}}\alpha_i)][{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}] + [S({}^{\text{ref}}\omega_i)][S({}^{\text{ref}}\omega_i)][{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}] + 2[S({}^{\text{ref}}\omega_i)][{}^{\text{ref}}_{i}R][{}^{i}v_{B/A}]$$

$$+ [{}^{\text{ref}}_{i}R][{}^{i}a_{B/A}]$$

Link Acceleration

$$[{}^{\text{ref}}a_B]$$

$$= [{}^{\text{ref}}a_A] + [S({}^{\text{ref}}\alpha_i)][{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}] + [S({}^{\text{ref}}\omega_i)][S({}^{\text{ref}}\omega_i)][{}^{\text{ref}}_{i}R][{}^{i}r_{B/A}]$$

$$+ 2[S({}^{\text{ref}}\omega_i)][{}^{\text{ref}}_{i}R][{}^{i}v_{B/A}] + [{}^{\text{ref}}_{i}R][{}^{i}a_{B/A}]$$

If both frames $\{\text{ref}\}$ and $\{i\}$ are coincident at this instant in time:

$$[{}^{i}a_B]$$

$$= [{}^{i}a_A] + [S({}^{i}\alpha_i)][{}^{i}_{i}R][{}^{i}r_{B/A}] + [S({}^{i}\omega_i)][S({}^{i}\omega_i)][{}^{i}_{i}R][{}^{i}r_{B/A}] + 2[S({}^{i}\omega_i)][{}^{i}_{i}R][{}^{i}v_{B/A}]$$

$$+ [{}^{i}_{i}R][{}^{i}a_{B/A}]$$

Note: $[{}^{i}_{i}R] = [I]$

$$[{}^{i}a_B]$$

$$= [{}^{i}a_A] + [S({}^{i}\alpha_i)][{}^{i}r_{B/A}] + [S({}^{i}\omega_i)][S({}^{i}\omega_i)][{}^{i}r_{B/A}] + 2[S({}^{i}\omega_i)][{}^{i}v_{B/A}] + [{}^{i}a_{B/A}]$$

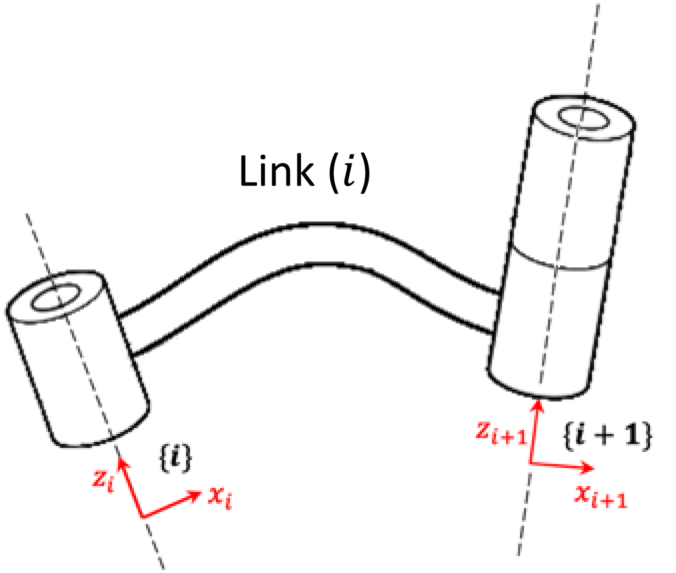
Link Acceleration

$$\left[{}^{i}a_{B}\right]$$

$$= \left[{}^{i}a_{A}\right] + \left[S({}^{i}\alpha_{i})\right]\left[{}^{i}r_{B/A}\right] + \left[S({}^{i}\omega_{i})\right]\left[S({}^{i}\omega_{i})\right]\left[{}^{i}r_{B/A}\right] + 2\left[S({}^{i}\omega_{i})\right]\left[{}^{i}v_{B/A}\right]$$

$$+ \left[{}^{i}a_{B/A}\right]$$

Applied to a link in the robotic arm:



Particle (A) → origin of $\{i\}$

Particle (B) → origin of $\{i + 1\}$

$\left[{}^{i}r_{B/A}\right] \rightarrow \left[{}^{i}P_{i+1}\right]$

If joint $(i + 1)$ is a **revolute joint**: $\left[{}^{i}v_{B/A}\right] = 0$ and $\left[{}^{i}a_{B/A}\right] = 0$

$$\left[{}^{i}a_{i+1}\right] = \left[{}^{i}a_{i}\right] + \left[S({}^{i}\alpha_{i})\right]\left[{}^{i}P_{i+1}\right] + \left[S({}^{i}\omega_{i})\right]\left[S({}^{i}\omega_{i})\right]\left[{}^{i}P_{i+1}\right]$$

Link Acceleration

$$[\,{}^{i}a_{i+1}] = [\,{}^{i}a_i] + [\,S({}^{i}\alpha_i)][{}^{i}P_{i+1}] + [\,S({}^{i}\omega_i)][\,S({}^{i}\omega_i)][{}^{i}P_{i+1}]$$

Pre-multiply both sides by $[{}^{i+1}_{\ \ i}\boldsymbol{R}]$:

$$[{}^{i+1}_{\ \ i}\boldsymbol{R}]\,[\,{}^{i}a_{i+1}] = [{}^{i+1}_{\ \ i}\boldsymbol{R}]([\,{}^{i}a_i] + [\,S({}^{i}\alpha_i)][{}^{i}P_{i+1}] + [\,S({}^{i}\omega_i)][\,S({}^{i}\omega_i)][{}^{i}P_{i+1}]\,)$$

$$\boxed{[{}^{i+1}a_{i+1}] = [{}^{i+1}_{\ \ i}R]([\,{}^{i}a_i] + [\,S({}^{i}\alpha_i)][{}^{i}P_{i+1}] + [\,S({}^{i}\omega_i)][\,S({}^{i}\omega_i)][{}^{i}P_{i+1}]\,)}$$

Link Acceleration

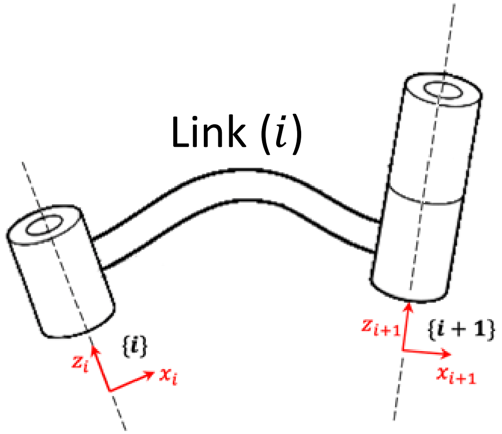
$$[\,{}^{i}a_B] = [\,{}^{i}a_A] + [S({}^{i}\alpha_i)][{}^{i}r_{B/A}] + [S({}^{i}\omega_i)][S({}^{i}\omega_i)][{}^{i}r_{B/A}] + 2[S({}^{i}\omega_i)][{}^{i}v_{B/A}] + [{}^{i}a_{B/A}]$$

Applied to a link in the robotic arm:

Particle (A) → origin of $\{i\}$

Particle (B) → origin of $\{i+1\}$

$[{}^{i}r_{B/A}] \rightarrow [\,{}^{i}P_{i+1}]$



If joint $(i+1)$ is a **prismatic joint**:

$$[{}^{i}v_{B/A}] = [{}_{i+1}^{\;\;i}R]\; {}^{i+1}\begin{bmatrix}0\\0\\\dot{d}\end{bmatrix} \quad \text{and} \quad [{}^{i}a_{B/A}] = [{}_{i+1}^{\;\;i}R]\; {}^{i+1}\begin{bmatrix}0\\0\\\ddot{d}\end{bmatrix}$$

$$[\,{}^{i}a_{i+1}] = [\,{}^{i}a_i] + [S({}^{i}\alpha_i)][{}^{i}P_{i+1}] + [S({}^{i}\omega_i)][S({}^{i}\omega_i)][{}^{i}P_{i+1}] + 2[S({}^{i}\omega_i)][{}_{i+1}^{\;\;i}R]\; {}^{i+1}\begin{bmatrix}0\\0\\\dot{d}\end{bmatrix} + [{}_{i+1}^{\;\;i}R]\; {}^{i+1}\begin{bmatrix}0\\0\\\ddot{d}\end{bmatrix}$$

Link Acceleration

$$[\,{}^{i}a_{i+1}] = [\,{}^{i}a_i] + [\,S({}^{i}\alpha_i)][{}^{i}P_{i+1}] + [\,S({}^{i}\omega_i)][\,S({}^{i}\omega_i)][{}^{i}P_{i+1}] + 2[\,S({}^{i}\omega_i)][{}_{i+1}^{\;\,i}R]\;{}^{i+1}\begin{bmatrix}0\\0\\\dot{d}\end{bmatrix} + [{}_{i+1}^{\;\,i}R]\;{}^{i+1}\begin{bmatrix}0\\0\\\ddot{d}\end{bmatrix}$$

Pre-multiply both sides by $[{}_{i}^{i+1}\boldsymbol{R}]$:

$$[{}_{i}^{i+1}\boldsymbol{R}]\,[\,{}^{i}a_{i+1}]$$

$$= [{}_{i}^{i+1}\boldsymbol{R}]([\,{}^{i}a_i] + [\,S({}^{i}\alpha_i)][{}^{i}P_{i+1}] + [\,S({}^{i}\omega_i)][\,S({}^{i}\omega_i)][{}^{i}P_{i+1}]) + 2[{}_{i}^{i+1}\boldsymbol{R}][\,S({}^{i}\omega_i)][{}_{i+1}^{\;\,i}R]\;{}^{i+1}\begin{bmatrix}0\\0\\\dot{d}\end{bmatrix}$$

$$+ [{}_{i}^{i+1}\boldsymbol{R}][{}_{i+1}^{\;\,i}R]\;{}^{i+1}\begin{bmatrix}0\\0\\\ddot{d}\end{bmatrix}$$

$$[{}^{i+1}a_{i+1}] = [{}_{i}^{i+1}\boldsymbol{R}]([\,{}^{i}a_i] + [\,S({}^{i}\alpha_i)][{}^{i}P_{i+1}] + [\,S({}^{i}\omega_i)][\,S({}^{i}\omega_i)][{}^{i}P_{i+1}]) + 2[{}_{i}^{i+1}\boldsymbol{R}][\,S({}^{i}\omega_i)][{}_{i+1}^{\;\,i}R]\;{}^{i+1}\begin{bmatrix}0\\0\\\dot{d}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot{d}\end{bmatrix}$$

Link Acceleration

Summary of linear acceleration:

If joint $(i+1)$ is a revolute joint:

$$\left[{}^{i+1}a_{i+1}\right] = \left[{}^{i+1}_{\ \ i}R\right]\left(\left[\,{}^{i}a_i\right] + \left[\,S({}^{i}\alpha_i)\right]\left[{}^{i}P_{i+1}\right] + \left[\,S({}^{i}\omega_i)\right]\left[\,S({}^{i}\omega_i)\right]\left[{}^{i}P_{i+1}\right]\right)$$

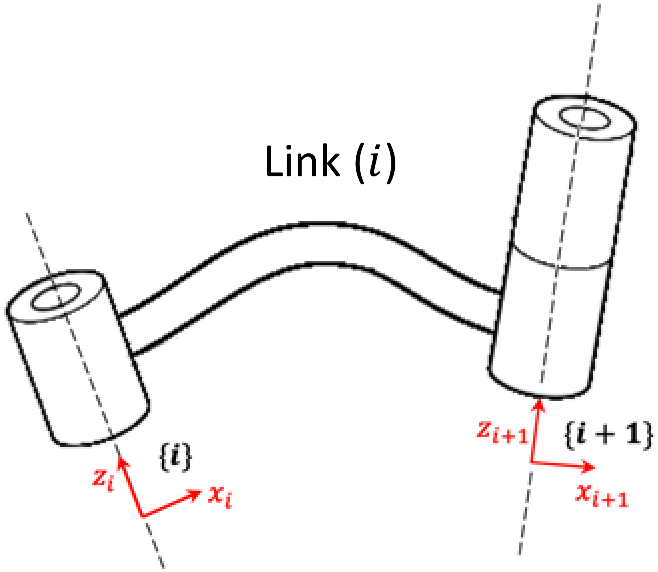
If joint $(i+1)$ is a prismatic joint:

$$\left[{}^{i+1}a_{i+1}\right] = \left[{}^{i+1}_{\ \ i}R\right]\left(\left[\,{}^{i}a_i\right] + \left[\,S({}^{i}\alpha_i)\right]\left[{}^{i}P_{i+1}\right] + \left[\,S({}^{i}\omega_i)\right]\left[\,S({}^{i}\omega_i)\right]\left[{}^{i}P_{i+1}\right]\right) + 2\left[{}^{i+1}_{\ \ i}R\right]\left[\,S({}^{i}\omega_i)\right]\left[{}^{\ \ i}_{i+1}R\right]{}^{i+1}\begin{bmatrix}0\\0\\\dot{d}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot{d}\end{bmatrix}$$

Link Acceleration

Angular acceleration:

If joint $(i+1)$ is a **revolute joint**:



$$\begin{bmatrix} \text{angular} \\ \text{velocity} \\ \text{of link (i + 1)} \end{bmatrix} = \begin{bmatrix} \text{angular} \\ \text{velocity} \\ \text{of link (i)} \end{bmatrix} + \begin{bmatrix} \text{the rotational} \\ \text{velocity of} \\ \text{joint (i + 1)} \end{bmatrix}$$

$$[\,{}^{i}\omega_{i+1}] = [\,{}^{i}\omega_i] + [{}_{i+1}^{\;\;i}R]\,{}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

Take the derivative of both sides with respect to time:

$$[\,{}^{i}\alpha_{i+1}] = [\,{}^{i}\alpha_i] + [\,S({}^{i}\omega_{i+1})][{}_{i+1}^{\;\;i}R]\,{}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} + [{}_{i+1}^{\;\;i}R]\,{}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{bmatrix}$$

Link Acceleration

$$[\,{}^{i}\alpha_{i+1}] = [\,{}^{i}\alpha_i] + [\,S({}^{i}\omega_{i+1})][{}_{i+1}^{\;\;i}R]\;{}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix} + [{}_{i+1}^{\;\;i}R]\;{}^{i+1}\begin{bmatrix}0\\0\\\ddot{\theta}_{i+1}\end{bmatrix}$$

Pre-multiply both sides by $[{}_{\;i}^{i+1}R]$:

$$[{}_{\;i}^{i+1}R]\,[\,{}^{i}\alpha_{i+1}] = [{}_{\;i}^{i+1}R][\,{}^{i}\alpha_i] + [{}_{\;i}^{i+1}R][\,S({}^{i}\omega_{i+1})][{}_{i+1}^{\;\;i}R]\;{}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix} + [{}_{\;i}^{i+1}R][{}_{i+1}^{\;\;i}R]\;{}^{i+1}\begin{bmatrix}0\\0\\\ddot{\theta}_{i+1}\end{bmatrix}$$

$$[{}^{i+1}\alpha_{i+1}] = [{}_{\;i}^{i+1}R][\,{}^{i}\alpha_i] + [{}_{\;i}^{i+1}R][\,S({}^{i}\omega_{i+1})][{}_{i+1}^{\;\;i}R]\;{}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot{\theta}_{i+1}\end{bmatrix}$$

Link Acceleration

$$\left[{}^{i+1}\alpha_{i+1}\right] = \left[{}^{i+1}_{\ i}R\right]\left[{}^{i}\alpha_i\right] + \left[{}^{i+1}_{\ i}R\right]\left[S({}^{i}\omega_{i+1})\right]\left[{}^{\ i}_{i+1}R\right]{}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot{\theta}_{i+1}\end{bmatrix}$$

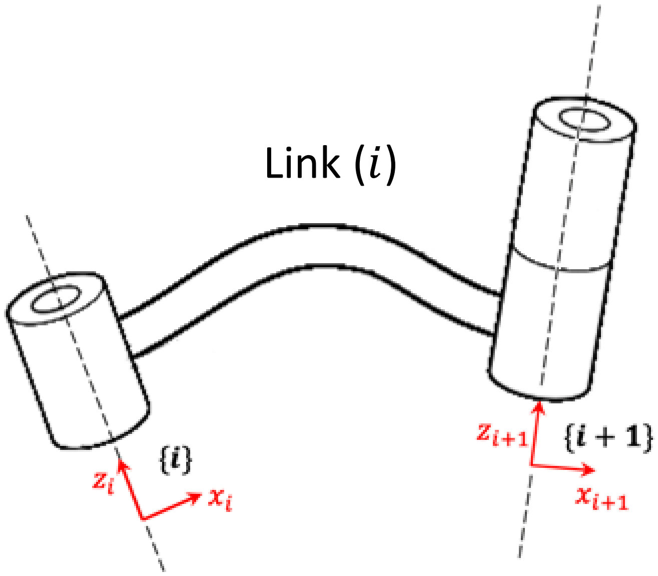
$$[R][S(a)][R]^T = [S([R][a])]$$

$$\left[{}^{i+1}\alpha_{i+1}\right] = \left[{}^{i+1}_{\ i}R\right]\left[{}^{i}\alpha_i\right] + \left[S\left(\left[{}^{i+1}_{\ i}R\right]\left[{}^{i}\omega_{i+1}\right]\right)\right]{}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot{\theta}_{i+1}\end{bmatrix}$$

$$\boxed{\left[{}^{i+1}\alpha_{i+1}\right] = \left[{}^{i+1}_{\ i}R\right]\left[{}^{i}\alpha_i\right] + \left[S({}^{i+1}\omega_{i+1})\right]{}^{i+1}\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix} + {}^{i+1}\begin{bmatrix}0\\0\\\ddot{\theta}_{i+1}\end{bmatrix}}$$

Link Acceleration

If joint $(i+1)$ is a prismatic joint:



$$\begin{bmatrix} \text{angular velocity} \\ \text{of link (i + 1)} \end{bmatrix} = \begin{bmatrix} \text{angular velocity} \\ \text{of link (i)} \end{bmatrix}$$

$$\left[\ {}^{i}\omega_{i+1}\right] = \left[\ {}^{i}\omega_{i}\right]$$

Take the derivative of both sides with respect to time:

$$\left[\ {}^{i}\alpha_{i+1}\right] = \left[\ {}^{i}\alpha_{i}\right]$$

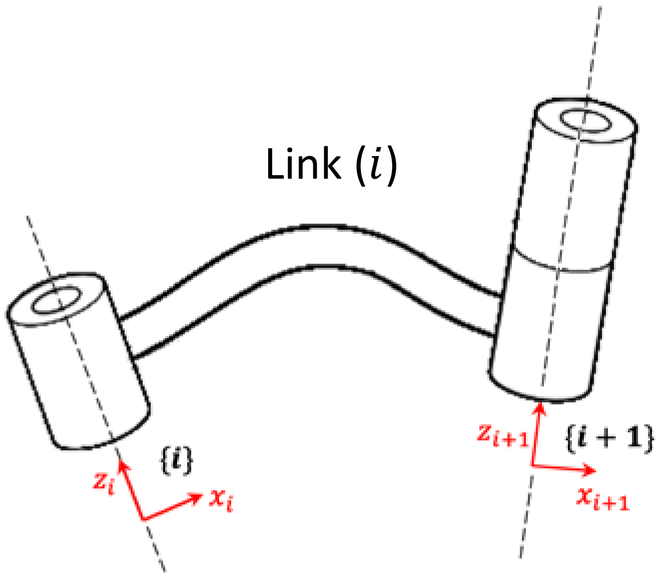
Link Acceleration

$$[\,{}^{i}\alpha_{i+1}] = [\,{}^{i}\alpha_{i}]$$

Pre-multiply both sides by $[{}^{i+1}_{\ i}R]$:

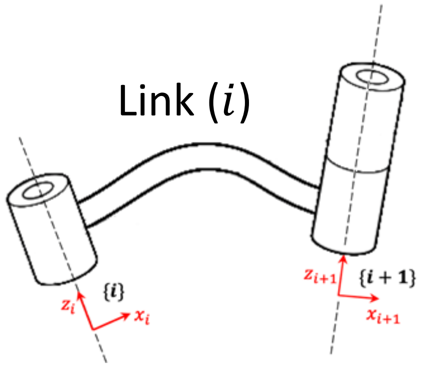
$$[{}^{i+1}_{\ i}R][{}^{i}\alpha_{i+1}] = [{}^{i+1}_{\ i}R][{}^{i}\alpha_{i}]$$

$$\boxed{[{}^{i+1}\alpha_{i+1}] = [{}^{i+1}_{\ i}R][{}^{i}\alpha_{i}]}$$



Link Acceleration

Summary of angular acceleration:



If joint $(i+1)$ is a revolute joint:

$$\left[{}^{i+1}\alpha_{i+1}\right] = \left[{}^{i+1}_{\ i}R\right]\left[{}^{i}\alpha_i\right] + \left[S\left({}^{i+1}\omega_{i+1}\right)\right]{}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} + {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{bmatrix}$$

If joint $(i+1)$ is a prismatic joint:

$$\left[{}^{i+1}\alpha_{i+1}\right] = \left[{}^{i+1}_{\ i}R\right]\left[{}^{i}\alpha_i\right]$$

Link Acceleration

For more details on this subject, please see:

- Introduction to Robotics: Mechanics and Control, by John J. Craig, 3rd Edition, Addison-Wesley Publishing Company, 2003.