Distance Learning Initiative

Introduction to Robotics

The Jacobian Matrix

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2020

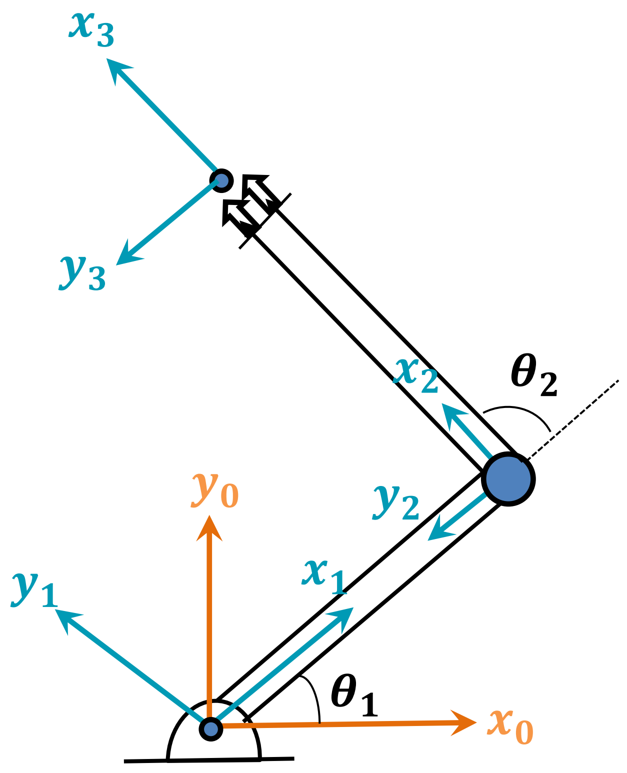
Jacobian Matrix

Example: For the planar 2 DOF RR robotic arm, calculate the velocity of each link and that of the end-effector as a function of the joint rates?

$$\left[\,{}^{1}\omega_1\right] = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \; \left[\,{}^{1}v_1\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\,{}^{2}\omega_2\right] = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}, \; \left[\,{}^{2}v_2\right] = \begin{bmatrix} \dot{\theta}_1 L_1 s_2 \\ \dot{\theta}_1 L_1 c_2 \\ 0 \end{bmatrix}$$

$$\left[\,{}^{3}\omega_3\right] = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}, \; \left[\,{}^{3}v_3\right] = \begin{bmatrix} \dot{\theta}_1 L_1 s_2 \\ \dot{\theta}_1 L_1 c_2 + (\dot{\theta}_1 + \dot{\theta}_2) L_2 \\ 0 \end{bmatrix}$$

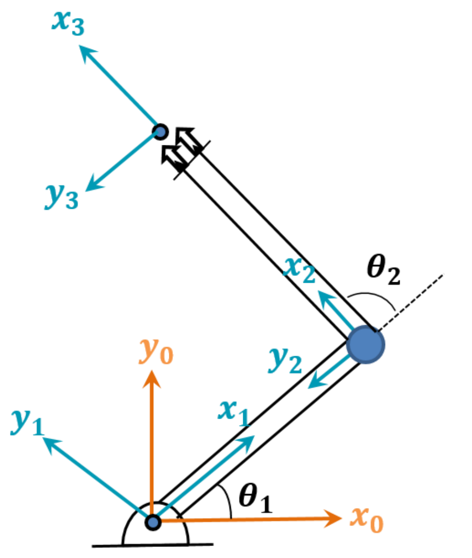


Jacobian Matrix

$$[\,^{3}\omega_3] = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix},\ [\,^{3}v_3] = \begin{bmatrix} \dot{\theta}_1 L_1 s_2 \\ \dot{\theta}_1 L_1 c_2 + (\dot{\theta}_1 + \dot{\theta}_2) L_2 \\ 0 \end{bmatrix}$$

$$[\,^{3}\omega_3] = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$[\,^{3}v_3] = \begin{bmatrix} \dot{\theta}_1 L_1 s_2 \\ \dot{\theta}_1 L_1 c_2 + (\dot{\theta}_1 + \dot{\theta}_2) l_2 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$



Jacobian Matrix

$$[\,{}^{3}\omega_3] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

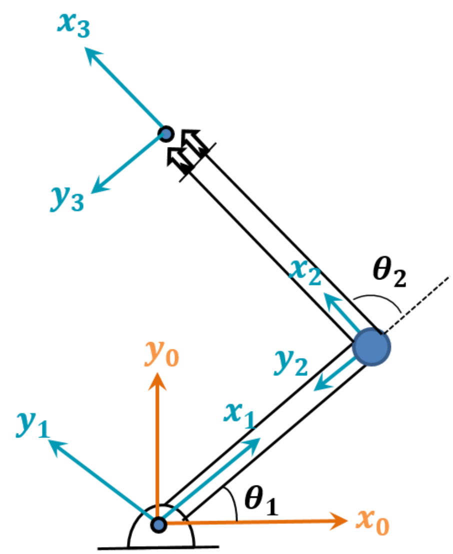
$[\mathbf{J}_\omega]$

$$[\,{}^{3}v_3] = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$[\mathbf{J}_v]$

$$^{3}\begin{bmatrix} [v_3] \\ [\omega_3] \end{bmatrix} = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$[\mathbf{J}]$



In robotics the Jacobian matrix [J] relates the end-effector velocity to the joint rates.

Jacobian Matrix

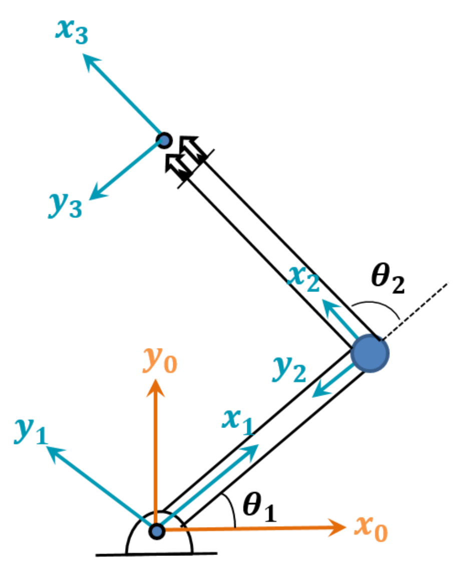
$$\left[{}^{3}v_3\right] = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$${}^{3}[\mathbf{J}_v]$$

$${}^{3}\begin{bmatrix} \dot{x}_3 \\ \dot{y}_3 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$${}^{3}[\mathbf{J}_v]_{2\times 2}$$

$${}^{3}\begin{bmatrix} \dot{x}_3 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$



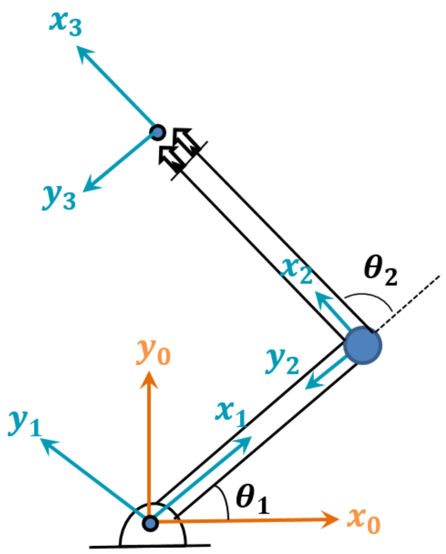
Jacobian Matrix

$${}^{3}[\mathrm{J}_v]$$

$$[\,{}^{3}v_3] = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$${}^{3}[\mathrm{J}_v] = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix}$$

$${}^{0}[\mathrm{J}_v] = [{}^{0}_{3}R]\ {}^{3}[\mathrm{J}_v]$$



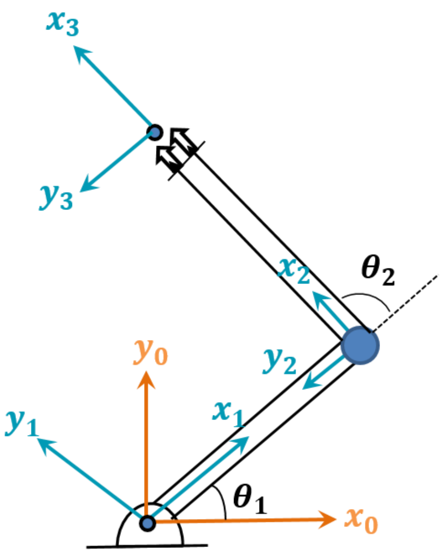
Jacobian Matrix

$$[{}^0_3R] = [{}^0_1R][{}^1_2R][{}^2_3R]$$

$$[{}^0_3R] = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^0_3R] = \begin{bmatrix} (c_1c_2 - s_1s_2) & (-c_1s_2 - s_1c_2) & 0 \\ (s_1c_2 + c_1s_2) & (-s_1s_2 + c_1c_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^0_3R] = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Jacobian Matrix

$$^{3}[\mathrm{J}_v]$$

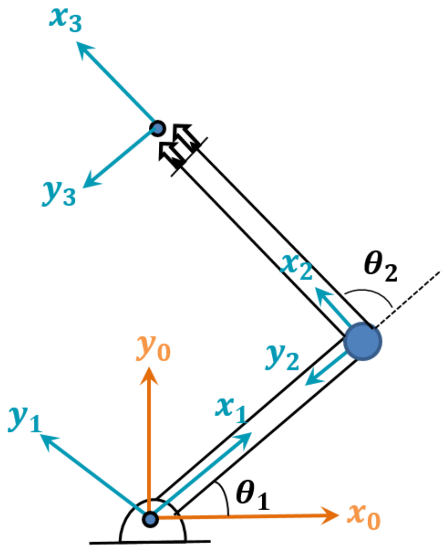
$$[\,^{3}v_3] = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$^{3}[\mathrm{J}_v] = \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix}$$

$$^{0}[\mathrm{J}_v] = [\,^{0}_{3}R\,]\;^{3}[\mathrm{J}_v]$$

$$^{0}[\mathrm{J}_v] = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_1 s_2 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix}$$

$$^{0}[\mathrm{J}_v] = \begin{bmatrix} -(L_1 s_1 + L_2 s_{12}) & -L_2 s_{12} \\ (L_1 c_1 + L_2 c_{12}) & L_2 c_{12} \\ 0 & 0 \end{bmatrix}$$



Jacobian Matrix

$$^0[\mathrm{J}_v] = \begin{bmatrix} -(L_1 s_1 + L_2 s_{12}) & -L_2 s_{12} \\ (L_1 c_1 + L_2 c_{12}) & L_2 c_{12} \\ 0 & 0 \end{bmatrix}$$

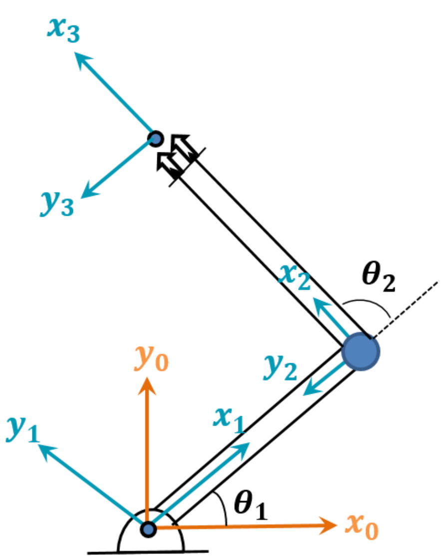
$$[\,^0v_3] = \begin{bmatrix} -(L_1 s_1 + L_2 s_{12}) & -L_2 s_{12} \\ (L_1 c_1 + L_2 c_{12}) & L_2 c_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$^0[\mathrm{J}_v]$$

$$^0\begin{bmatrix} \dot{x}_3 \\ \dot{y}_3 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -(L_1 s_1 + L_2 s_{12}) & -L_2 s_{12} \\ (L_1 c_1 + L_2 c_{12}) & L_2 c_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$^0\begin{bmatrix} \dot{x}_3 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} -(L_1 s_1 + L_2 s_{12}) & -L_2 s_{12} \\ (L_1 c_1 + L_2 c_{12}) & L_2 c_{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$^0[\mathrm{J}_v]_{2\times 2}$$



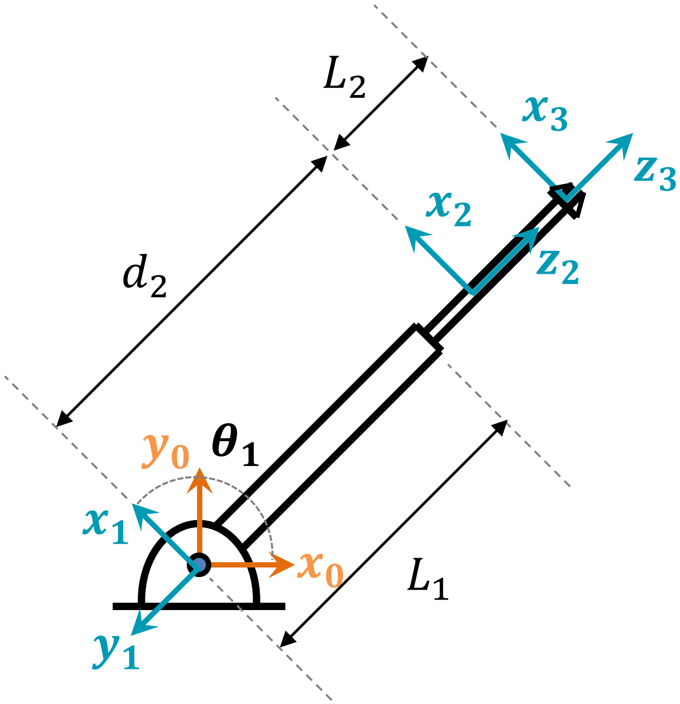
Jacobian Matrix

Example: For the planar 2 DOF RP robotic arm, find the Jacobian matrix?

$$[\,{}^{1}\omega_1] = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \;;\; [\,{}^{1}v_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[\,{}^{2}\omega_2] = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \;;\; [\,{}^{2}v_2] = \begin{bmatrix} \dot{\theta}_1 d_2 \\ 0 \\ \dot{d}_2 \end{bmatrix}$$

$$[\,{}^{3}\omega_3] = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \;;\; [\,{}^{3}v_3] = \begin{bmatrix} \dot{\theta}_1 (d_2 + L_2) \\ 0 \\ \dot{d}_2 \end{bmatrix}$$

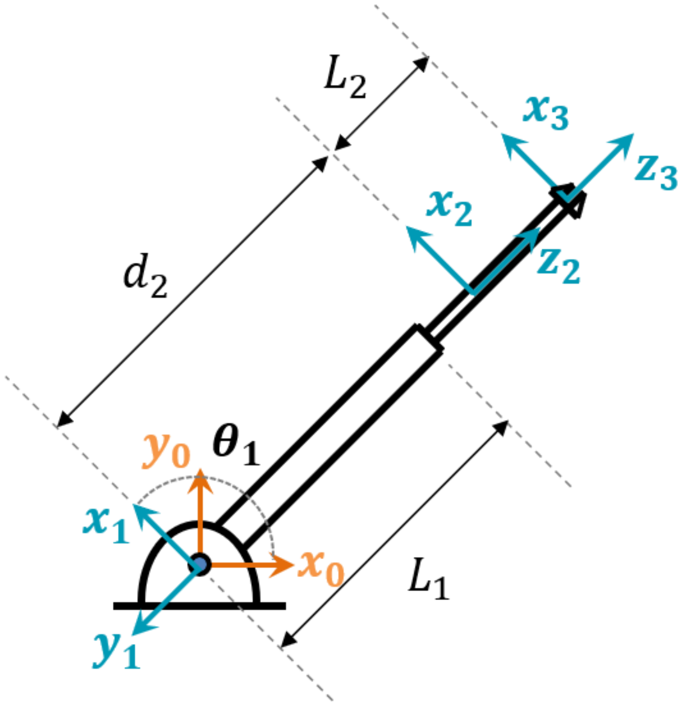


Jacobian Matrix

$$[\,^3\omega_3] = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \quad ; \quad [\,^3v_3] = \begin{bmatrix} \dot{\theta}_1(d_2 + L_2) \\ 0 \\ \dot{d}_2 \end{bmatrix}$$

$$[\,^3\omega_3] = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

$$[\,^3v_3] = \begin{bmatrix} \dot{\theta}_1(d_2 + L_2) \\ 0 \\ \dot{d}_2 \end{bmatrix} = \begin{bmatrix} (d_2 + L_2) & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

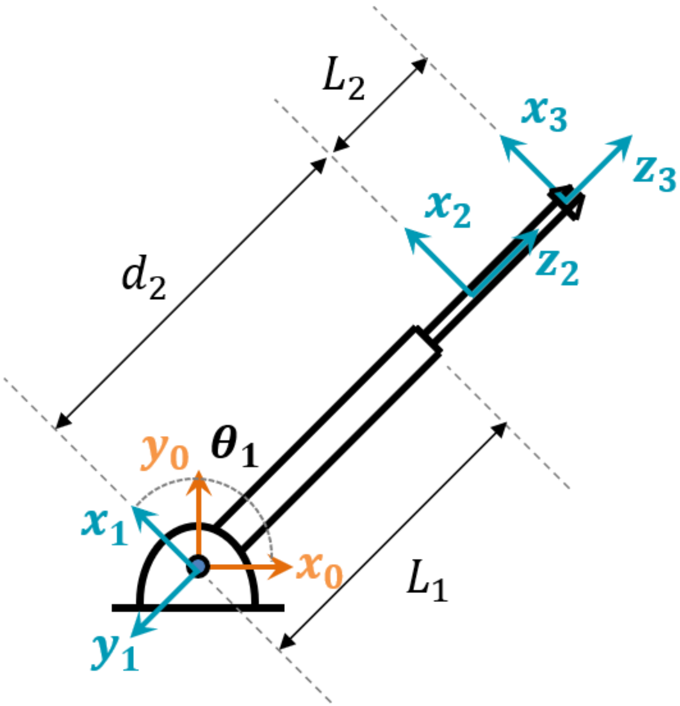


Jacobian Matrix

$$[{}^{3}\omega_3] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix} \Rightarrow [\mathbf{J}_\omega]$$

$$[{}^{3}v_3] = \begin{bmatrix} (d_2 + L_2) & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix} \Rightarrow [\mathbf{J}_v]$$

$${}^{3}\begin{bmatrix} [v_3] \\ [\omega_3] \end{bmatrix} = \begin{bmatrix} (d_2 + L_2) & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix} \Rightarrow [\mathbf{J}]$$



Jacobian Matrix

$${}^{3}[\mathrm{J}_v]$$

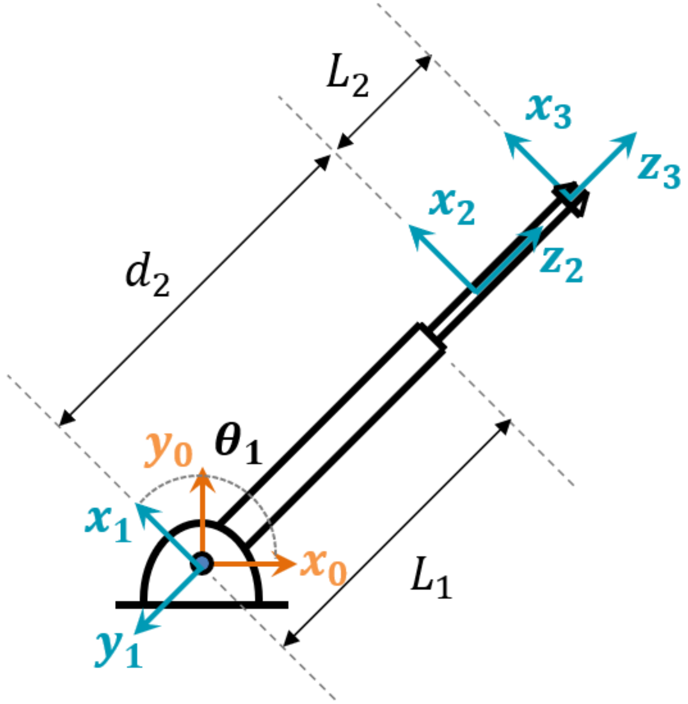
$$[\,{}^{3}v_3] = \begin{bmatrix} (d_2 + L_2) & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

$${}^{3}[\mathrm{J}_v] = \begin{bmatrix} (d_2 + L_2) & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$${}^{0}[\mathrm{J}_v] = [{}^{0}_{3}R]\;{}^{3}[\mathrm{J}_v]$$

$${}^{0}[\mathrm{J}_v] = \begin{bmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (d_2 + L_2) & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$${}^{0}[\mathrm{J}_v] = \begin{bmatrix} c_1(d_2 + L_2) & s_1 \\ s_1(d_2 + L_2) & -c_1 \\ 0 & 0 \end{bmatrix}$$



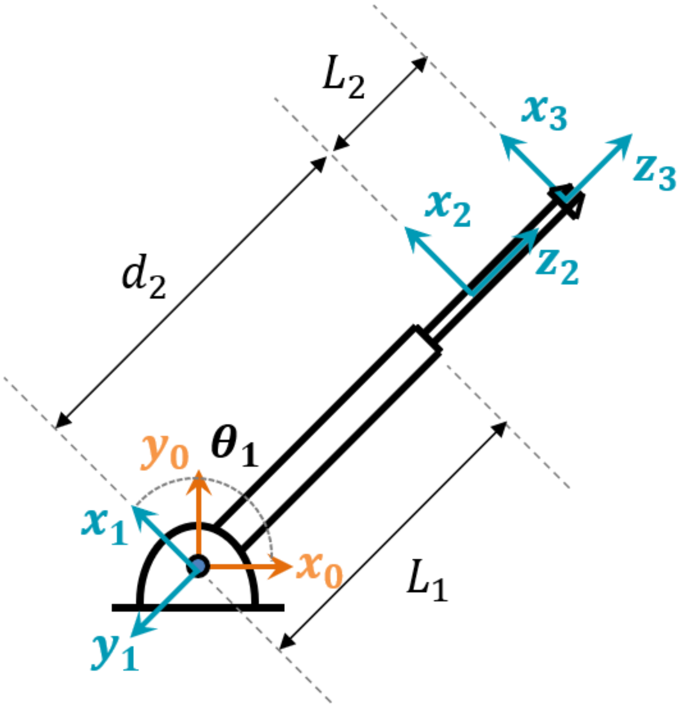
Jacobian Matrix

$$^{0}[\mathrm{J}_v] = \begin{bmatrix} c_1(d_2+L_2) & s_1 \\ s_1(d_2+L_2) & -c_1 \\ 0 & 0 \end{bmatrix} \Rightarrow {}^{0}[\mathrm{J}_v]$$

$$[\,^{0}v_3] = \begin{bmatrix} c_1(d_2+L_2) & s_1 \\ s_1(d_2+L_2) & -c_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

$$^{0}\begin{bmatrix} \dot{x}_3 \\ \dot{y}_3 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} c_1(d_2+L_2) & s_1 \\ s_1(d_2+L_2) & -c_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

$$^{0}\begin{bmatrix} \dot{x}_3 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} c_1(d_2+L_2) & s_1 \\ s_1(d_2+L_2) & -c_1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix} \Rightarrow {}^{0}[\mathrm{J}_v]_{2\times 2}$$



Jacobian Matrix

For more details on this subject, please see:

- Introduction to Robotics: Mechanics and Control, by John J. Craig, 3rd Edition, Addison-Wesley Publishing Company, 2003.