

Distance Learning Initiative

Introduction to Robotics

Mass Moment of Inertia

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Mass Moment of Inertia

Mass Moment of Inertia:

- It measures the ability of an object to resist change in its rotational speed about a specific axis.
- It represents rotational inertia about a specific axis.
- It is a measure of the distribution of the mass of an object relative to a given axis.



Source: www.nasa.gov

Mass Moment of Inertia

The role of mass moment of inertia in rotational mechanics is similar to that of mass in linear mechanics.

	Linear	Angular
Newton's 2 nd Law	$F = ma$	$T = I\alpha$
Momentum	$p = mv$	$L = I\omega$
Kinetic Energy	$K.E. = \frac{1}{2}mv^2$	$K.E. = \frac{1}{2}I\omega^2$

Mass Moment of Inertia

For a point mass:

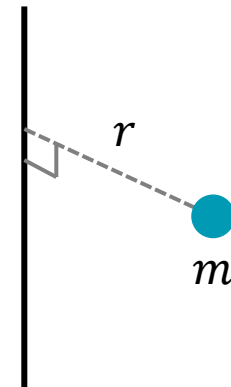
The mass moment of inertia is given by:

$$I = mr^2$$

m : is the mass of the particle

r : is distance from the reference axis

It has units of kg.m^2



Mass Moment of Inertia

$$I = mr^2$$

When a mass moves further from the axis of rotation, (I) becomes larger.

$$T = I\alpha$$

For a given torque, the larger the (I) of a body, the smaller the resulting angular acceleration.



Source: [The Walk 2015 \(IMDb\)](#)

Mass Moment of Inertia

Mass moment of inertia of a rigid body:

When calculating the mass moment of inertia for a rigid body, one thinks of the body as a sum of particles, each having a mass of dm (an infinitesimal or a differential mass).

$$dI = r^2 dm$$

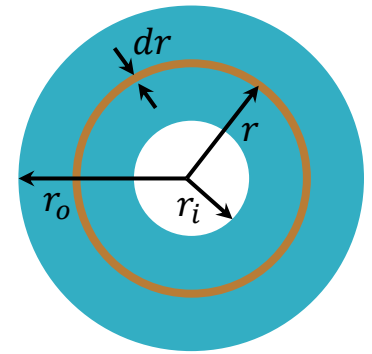
Then the mass moment of inertia of the body would be calculated as:

$$I = \int_0^M r^2 dm$$

Mass Moment of Inertia

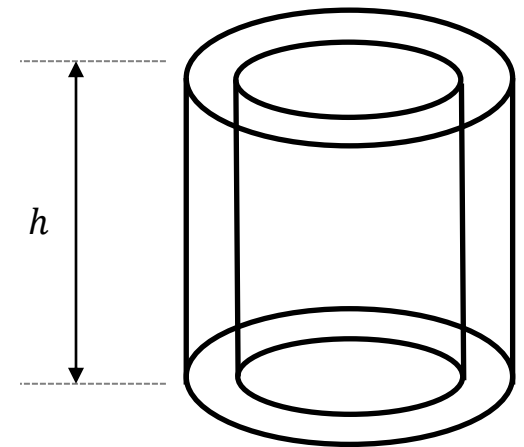
Example:

Derive the mass moment of inertia of a hollow cylinder. The cylinder has a mass (M), an inner radius (r_i), an outer radius (r_o), and a height (h).



The mass moment of inertia about a certain axis is:

$$I = \int_0^M r^2 dm$$



Mass Moment of Inertia

Consider a differential ring with a mass of (dm),

$$dm = \rho dV$$

where (ρ) is the density,

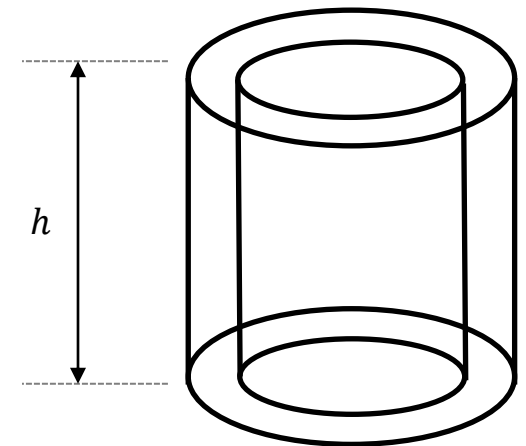
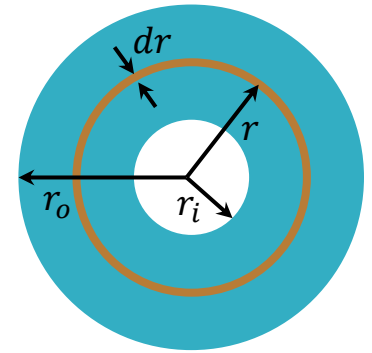
$$dV = dA \times h$$

where (dA) is the area of the differential ring,

$$dA = \pi(r + dr)^2 - \pi r^2$$

$$dA = \pi(r^2 + 2rdr + (dr)^2) - \pi r^2$$

$$dA = 2\pi r dr$$



Mass Moment of Inertia

$$dA = 2\pi r dr$$

Substitute (dA) into

$$dV = dA \times h$$

gives

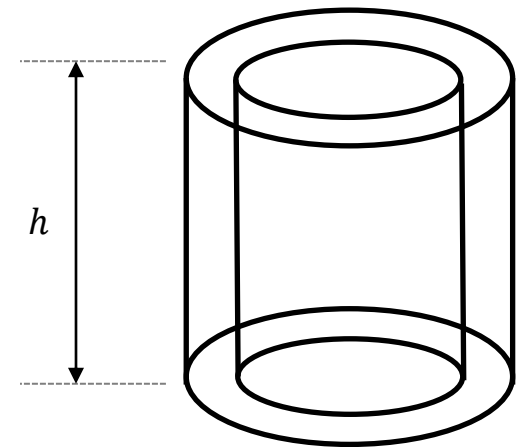
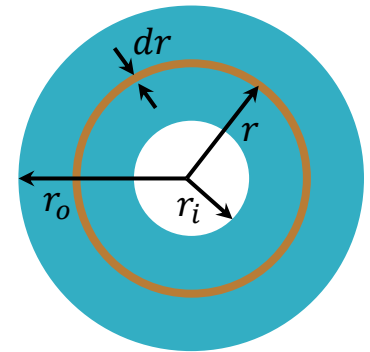
$$dV = 2\pi r h dr$$

Now substitute (dV) into

$$dm = \rho dV$$

To get

$$dm = \rho 2\pi r h dr$$



Mass Moment of Inertia

Substituting ($dm = \rho 2\pi r h dr$) into

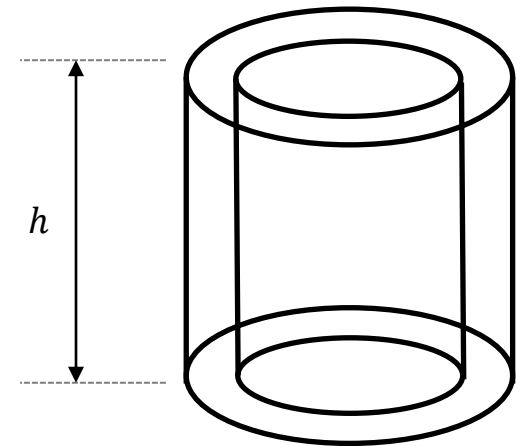
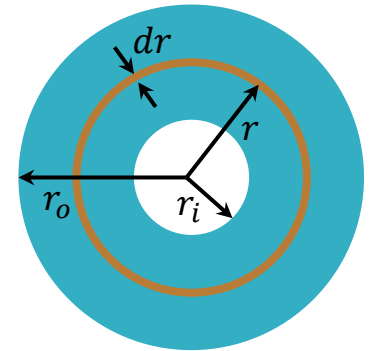
$$I = \int_0^M r^2 dm$$

$$I = \int_{r_i}^{r_o} 2\rho\pi r^3 h dr$$

$$I = 2\rho\pi h \int_{r_i}^{r_o} r^3 dr$$

$$I = 2\rho\pi h \left[\frac{r_o^4}{4} - \frac{r_i^4}{4} \right]$$

$$I = \frac{\rho\pi h}{2} [r_o^4 - r_i^4]$$



Mass Moment of Inertia

$$I = \frac{\rho\pi h}{2} [r_o^4 - r_i^4]$$

Note that density is defined as

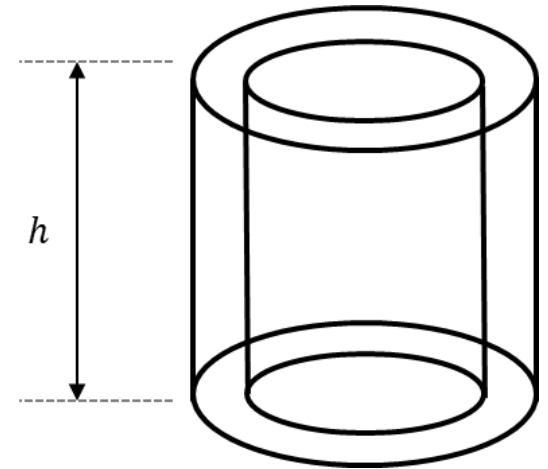
$$\rho = \frac{M}{V} = \frac{M}{(\pi r_o^2 - \pi r_i^2)h}$$

Substitute into the expression for (I)

$$I = \frac{M}{(\pi r_o^2 - \pi r_i^2)h} \frac{\pi h}{2} [r_o^4 - r_i^4]$$

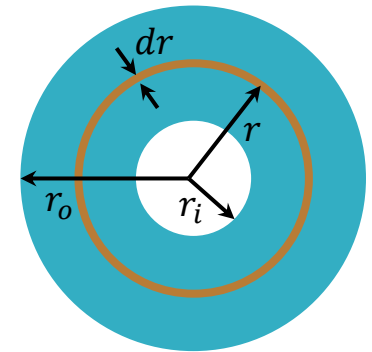
$$I = \frac{M}{2(r_o^2 - r_i^2)} [r_o^4 - r_i^4] = \frac{M}{2(r_o^2 - r_i^2)} (r_o^2 - r_i^2)(r_o^2 + r_i^2)$$

$$\therefore I = \frac{M}{2} (r_i^2 + r_o^2)$$



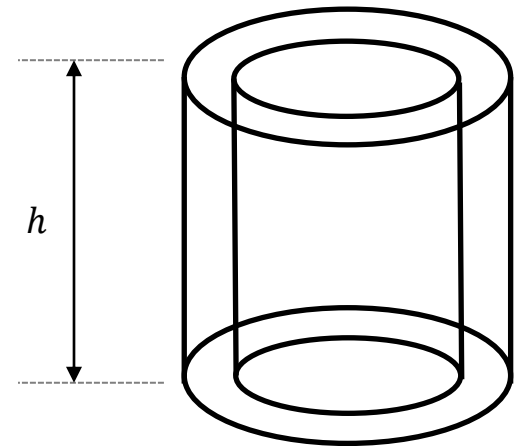
Mass Moment of Inertia

The mass moment of inertia of a hollow cylinder,



$$I = \frac{M}{2} (r_i^2 + r_o^2)$$

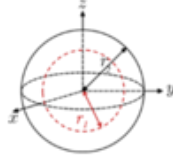
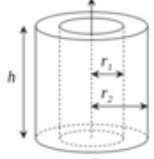
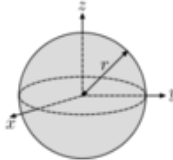
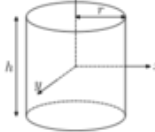
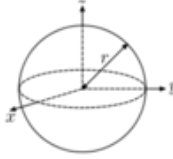
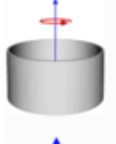
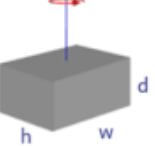

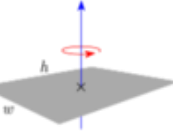
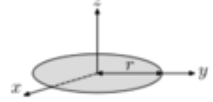
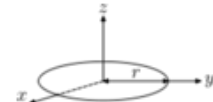
For a solid cylinder, $r_i = 0$, $r_o = r$, then



$$I = \frac{M}{2} r^2$$

Mass Moment of Inertia

For objects with simple geometrical shapes it is possible to express the mass moment of inertia in a mathematical form.

Thick-walled hollow sphere	 $I = \frac{2m}{5} \left(\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3} \right)$	Thick-walled hollow cylinder	 $I_z = \frac{1}{2} m(r_1^2 + r_2^2)$ $I_x = I_y = \frac{1}{12} m[3(r_1^2 + r_2^2) + h^2]$
Solid sphere	 $I = \frac{2}{5} mr^2$	Solid cylinder	 $I_z = \frac{1}{2} mr^2$
Hollow sphere	 $I = \frac{2}{3} mr^2$	Thin-walled hollow cylinder	 $I = mr^2$
Solid rectangular box	 $I_d = \frac{1}{12} m(h^2 + w^2)$	Thin rod	 $I_{center} = \frac{1}{12} mL^2$
Solid rectangular plate	 $I_{center} = \frac{1}{12} m(h^2 + w^2)$	Solid circular plate	 $I_z = \frac{1}{2} mr^2$ $I_x = I_y = \frac{1}{4} mr^2$
		Hollow plate (ring)	 $I_z = mr^2$ $I_x = I_y = \frac{1}{2} mr^2$

Source: <https://efcms.engr.utk.edu/ef151-2019-01/pilot/classmgr.php?c=43&p=mmi>

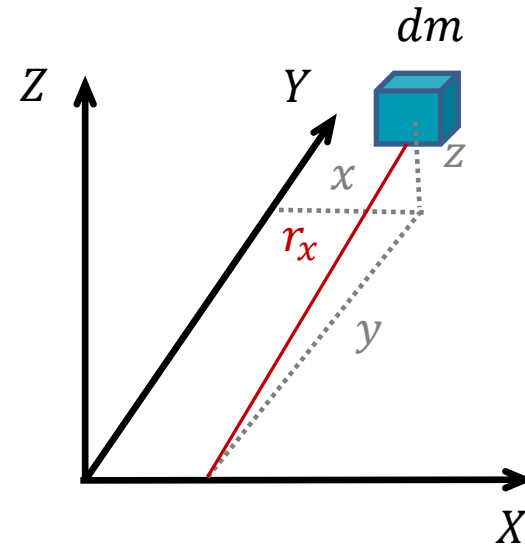
Mass Moment of Inertia

$$I = \int_0^M r^2 dm$$

$$I_x = \int r_x^2 dm = \int (y^2 + z^2) dm$$

$$I_y = \int r_y^2 dm = \int (z^2 + x^2) dm$$

$$I_z = \int r_z^2 dm = \int (x^2 + y^2) dm$$



x is the distance from the YZ -plane to dm .
 y is the distance from the ZX -plane to dm .
 z is the distance from the XY -plane to dm .

Mass Moment of Inertia

It has several names:

- 1- Moment of inertia.
- 2- Second moment of mass.
- 3- Angular mass.
- 4- Rotational inertia.

It should not be confused with the second moment of area!

Mass Moment of Inertia

Second moment of area:

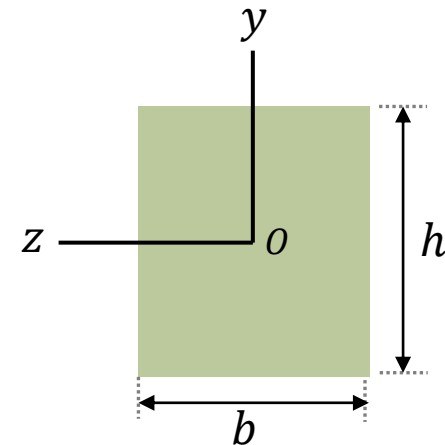
(the area moment of inertia):

$$I_Z = \int_A y^2 dA$$

For a rectangular cross section: $I_Z = \frac{bh^3}{12}$

For a circular cross section: $I_Z = \frac{\pi}{64} d^4$

$$\kappa = \frac{M}{EI_Z}$$



Mass Moment of Inertia

Mass moment of inertia	Second moment of area
$I_z = \int_0^M (y^2 + z^2) dm$	$I_z = \int_A y^2 dA$
Resistance to change in its rotational speed about a specific axis	Resistance to bending about a specific axis
A measure of the distribution of the mass of an object relative to a given axis.	A measure of the distribution of the area of a cross-section relative to a given axis.
Units: kg.m ²	Units: m ⁴

Mass Moment of Inertia

Parallel-axis theorem:

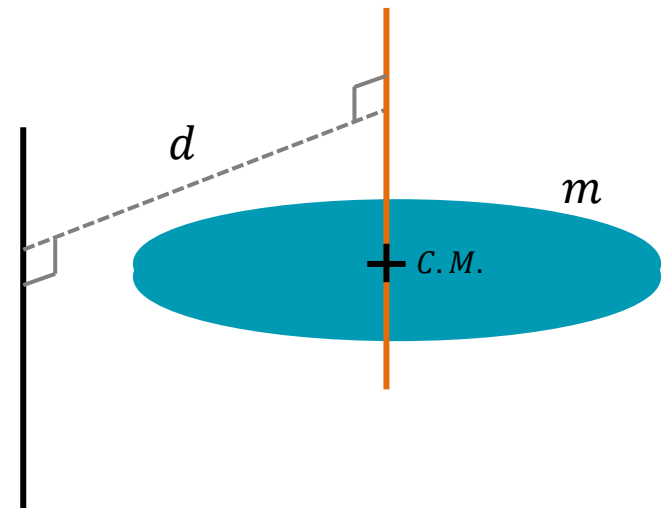
The moment of inertia about any axis can be calculated from the moment of inertia about a parallel axis that passes through the center of mass.

$$I = I_G + md^2$$

where:

d : is the distance between the two axes.

m : is the total mass of the body.



Mass Moment of Inertia

For an object rotating about a certain axis with an angular speed of (ω), the angular momentum can be found from:

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where

$$I_{xx} = \int (y^2 + z^2) dm \quad : \text{moment of inertia about the } x\text{-axis}$$

$$I_{yy} = \int (z^2 + x^2) dm \quad : \text{moment of inertia about the } y\text{-axis}$$

$$I_{zz} = \int (x^2 + y^2) dm \quad : \text{moment of inertia about the } z\text{-axis}$$

$$I_{xy} = I_{yx} = - \int xy dm \quad : xy \text{ product of inertia}$$

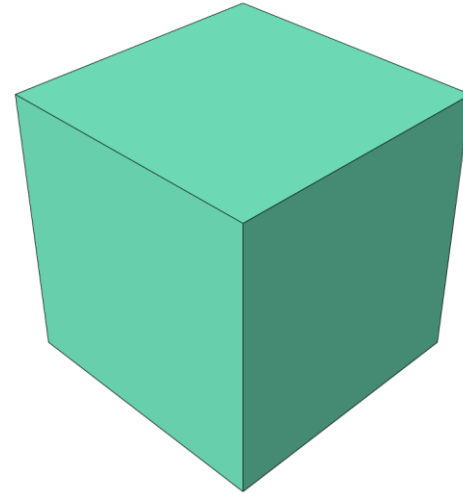
$$I_{yz} = I_{zy} = - \int yz dm \quad : yz \text{ product of inertia}$$

$$I_{xz} = I_{zx} = - \int xz dm \quad : xz \text{ product of inertia}$$

Mass Moment of Inertia

Example:

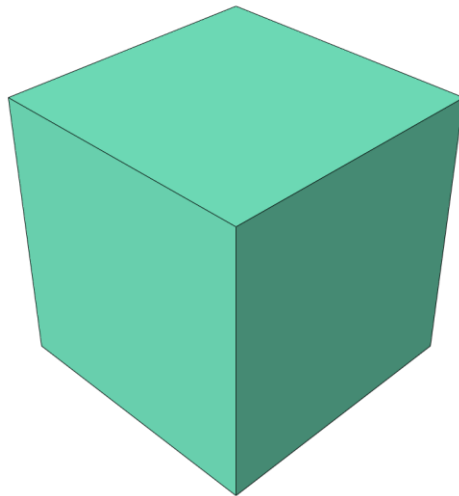
A cube of steel
(density of $7700 \frac{kg}{m^3}$)
with an edge of
length of 0.25 m.



$$I = \frac{1}{12} m(h^2 + w^2)$$

$$I = \frac{1}{12} \left(7700 \frac{kg}{m^3} \times (0.25^3) \right) (0.25^2 + 0.25^2) = 1.25325 \text{ kg} \cdot \text{m}^2$$

Mass Moment of Inertia



Mass properties:

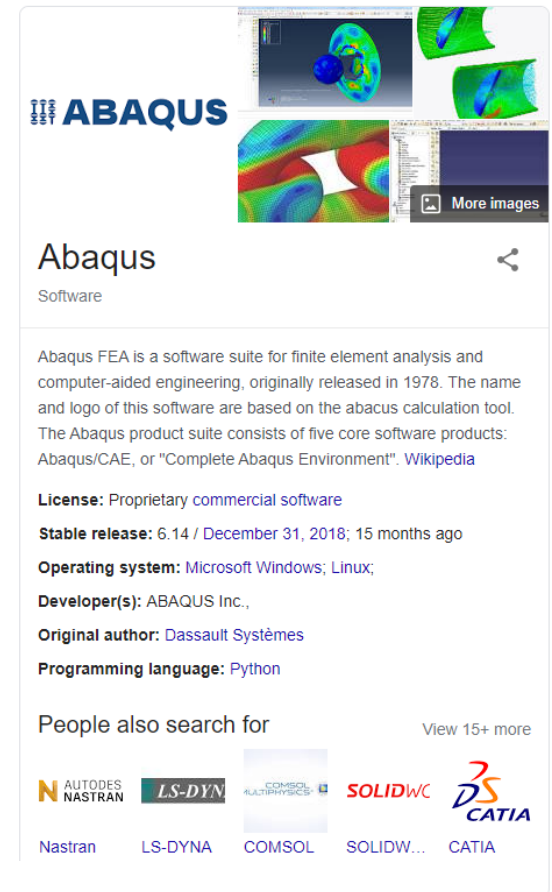
Volume: 0.0156

Volume centroid: -19.88, -19.88, 0.125

Mass: 120.31

Center of mass: -19.88, -19.88, 0.125

Moment of inertia about the center of mass (I_{xx} , I_{yy} , I_{zz} , I_{xy} , I_{yz} , I_{zx}): 1.25, 1.25, 1.25, 0, 0, 0



ABAQUS

Abaqus

Software

Abaqus FEA is a software suite for finite element analysis and computer-aided engineering, originally released in 1978. The name and logo of this software are based on the abacus calculation tool. The Abaqus product suite consists of five core software products: Abaqus/CAE, or "Complete Abaqus Environment". [Wikipedia](#)

License: Proprietary commercial software

Stable release: 6.14 / December 31, 2018; 15 months ago

Operating system: Microsoft Windows; Linux;

Developer(s): ABAQUS Inc.,

Original author: Dassault Systèmes

Programming language: Python

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