Distance Learning Initiative

Introduction to Robotics

Mass Moment of Inertia

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Mass Moment of Inertia:

- It measures the ability of an object to resist change in its rotational speed about a specific axis.
- It represents rotational inertia about a specific axis.
- It is a measure of the distribution of the mass of an object relative to a given axis.

Source: www.nasa.gov

Mass Moment of Inertia

The role of mass moment of inertia in rotational mechanics is similar to that of mass in linear mechanics.

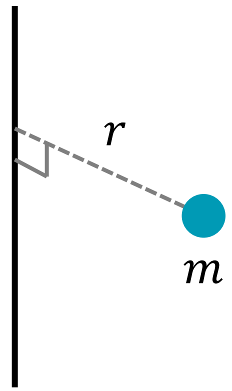
	Linear	Angular
Newton's 2nd Law	$F = ma$	$T = I\alpha$
Momentum	$p = mv$	$L = I\omega$
Kinetic Energy	$K.E. = \frac{1}{2}mv^2$	$K.E. = \frac{1}{2}I\omega^2$

Mass Moment of Inertia

For a point mass:

The mass moment of inertia is given by:

$$I = mr^2$$



m : is the mass of the particle

r : is distance from the reference axis

It has units of kg.m^2

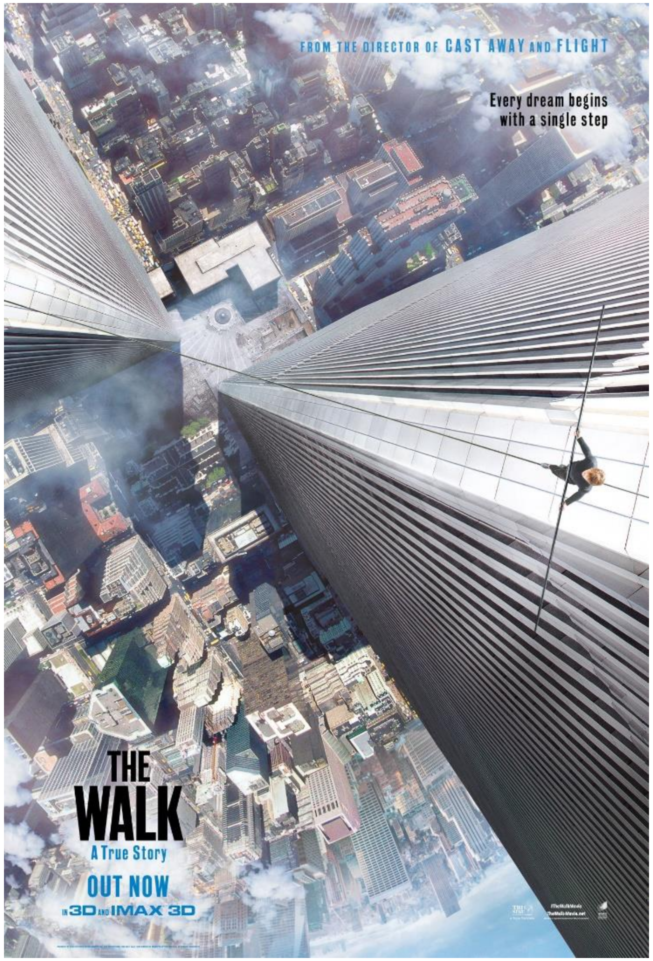
Mass Moment of Inertia

$I = mr^2$

When a mass moves further from the axis of rotation, (I) becomes larger.

$T = I\alpha$

For a given torque, the larger the (I) of a body, the smaller the resulting angular acceleration.



Source: The Walk 2015 (IMDb)

Mass moment of inertia of a rigid body:

When calculating the mass moment of inertia for a rigid body, one thinks of the body as a sum of particles, each having a mass of dm (an infinitesimal or a differential mass).

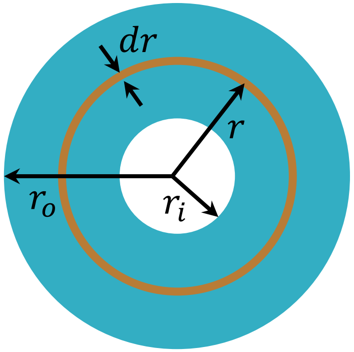
$$dI = r^2 dm$$

Then the mass moment of inertia of the body would be calculated as:

$$I = \int_0^M r^2 dm$$

Mass Moment of Inertia

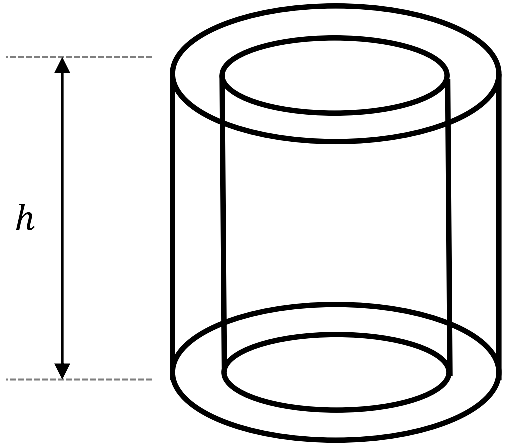
Example:



Derive the mass moment of inertia of a hollow cylinder. The cylinder has a mass (M), an inner radius (r_i), an outer radius (r_o), and a height (h).

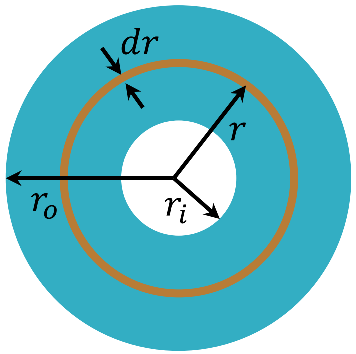
The mass moment of inertia about a certain axis is:

$$I = \int_0^M r^2 dm$$



Mass Moment of Inertia

Consider a differential ring with a mass of (dm),

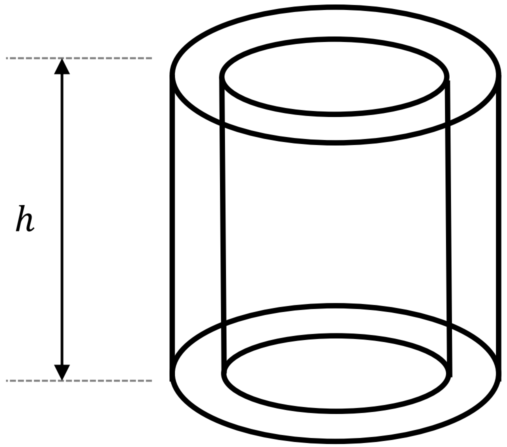


$$dm = \rho dV$$

where (ρ) is the density,

$$dV = dA \times h$$

where (dA) is the area of the differential ring,



$$dA = \pi(r + dr)^2 - \pi r^2$$

$$dA = \pi(r^2 + 2rdr + (dr)^2) - \pi r^2$$

$$dA = 2\pi rdr$$

Mass Moment of Inertia

$dA = 2\pi r dr$

Substitute (dA) into

$dV = dA \times h$

gives

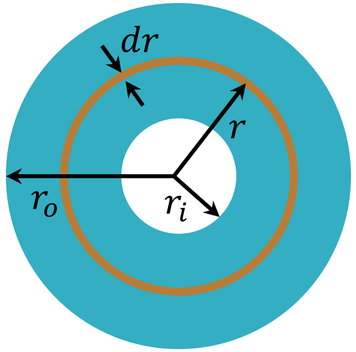
$dV = 2\pi r h dr$

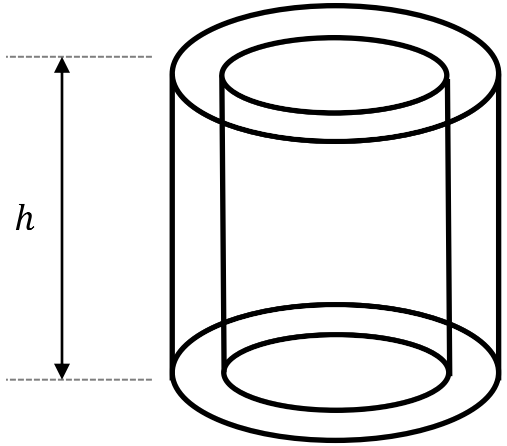
Now substitute (dV) into

$dm = \rho dV$

To get

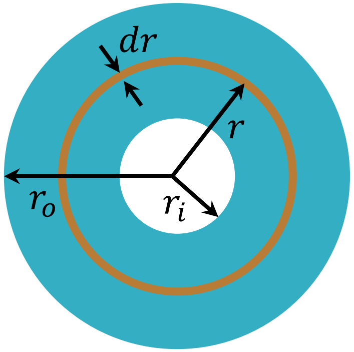
$dm = \rho 2\pi r h dr$





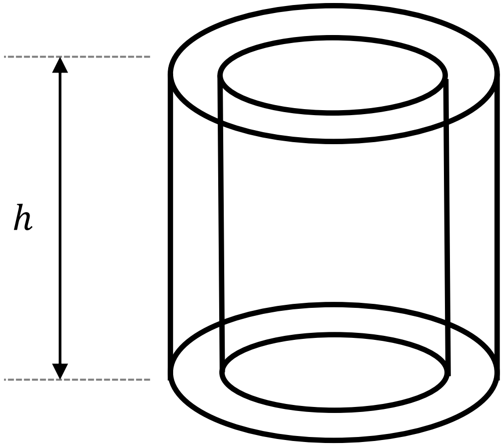
Mass Moment of Inertia

Substituting ($dm = \rho 2\pi r h dr$) into



$$I = \int_0^M r^2 dm$$

$$I = \int_{r_i}^{r_o} 2\rho\pi r^3 h dr$$



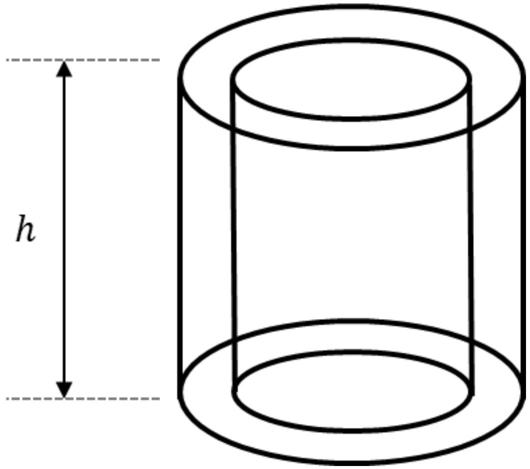
$$I = 2\rho\pi h \int_{r_i}^{r_o} r^3 dr$$

$$I = 2\rho\pi h \left[\frac{{r_o}^4}{4} - \frac{{r_i}^4}{4}\right]$$

$$I = \frac{\rho\pi h}{2}\left[{r_o}^4 - {r_i}^4\right]$$

Mass Moment of Inertia

$$I = \frac{\rho \pi h}{2}[r_o{}^4 - r_i{}^4]$$



Note that density is defined as

$$\rho = \frac{M}{V} = \frac{M}{(\pi r_o{}^2 - \pi r_i{}^2)h}$$

Substitute into the expression for (I)

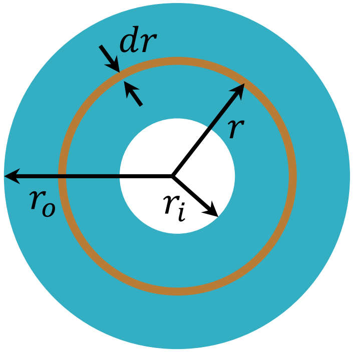
$$I = \frac{M}{(\pi r_o{}^2 - \pi r_i{}^2)h} \frac{\pi h}{2}[r_o{}^4 - r_i{}^4]$$

$$I = \frac{M}{2(r_o{}^2 - r_i{}^2)}[r_o{}^4 - r_i{}^4] = \frac{M}{2(r_o{}^2 - r_i{}^2)}(r_o{}^2 - r_i{}^2)(r_o{}^2 + r_i{}^2)$$

$$\therefore \; I = \frac{M}{2}(r_i{}^2 + r_o{}^2)$$

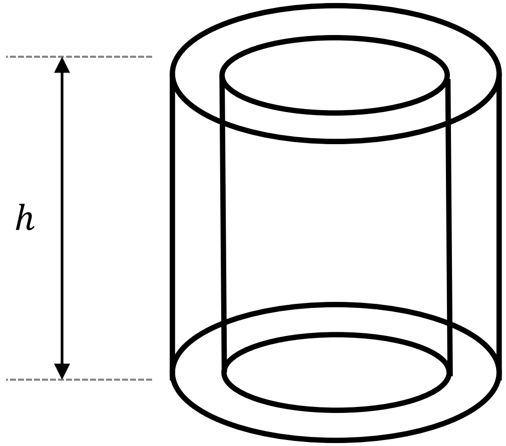
Mass Moment of Inertia

The mass moment of inertia of a hollow cylinder,



$$I = \frac{M}{2}({r_i}^2 + {r_o}^2)$$

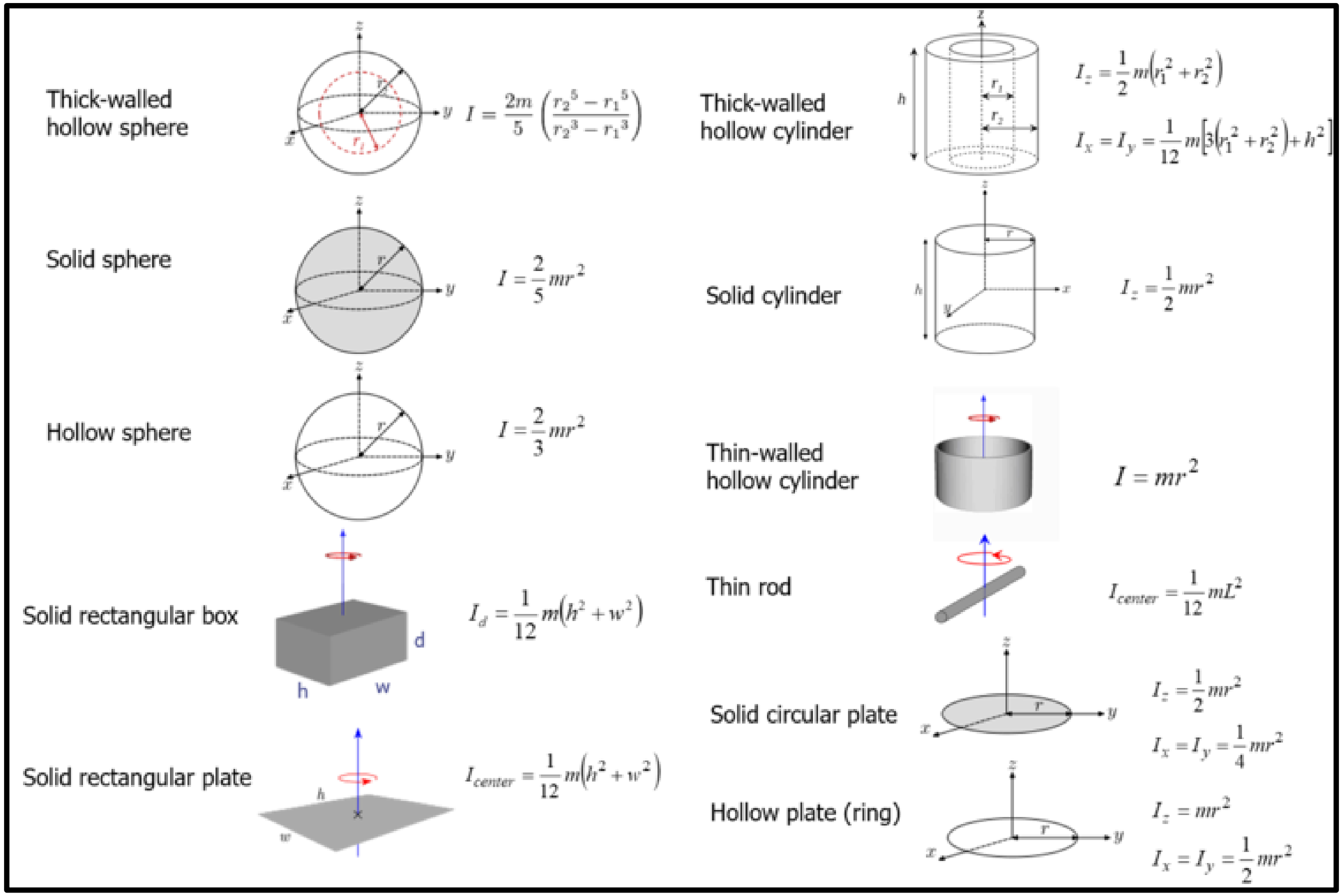
For a solid cylinder , $r_i = 0$, $r_o = r$, then



$$I = \frac{M}{2}r^2$$

Mass Moment of Inertia

For objects with simple geometrical shapes it is possible to express the mass moment of inertia in a mathematical form.



Source: https://efcms.engr.utk.edu/ef151-2019-01/pilot/classmgr.php?c=43&p=mmi

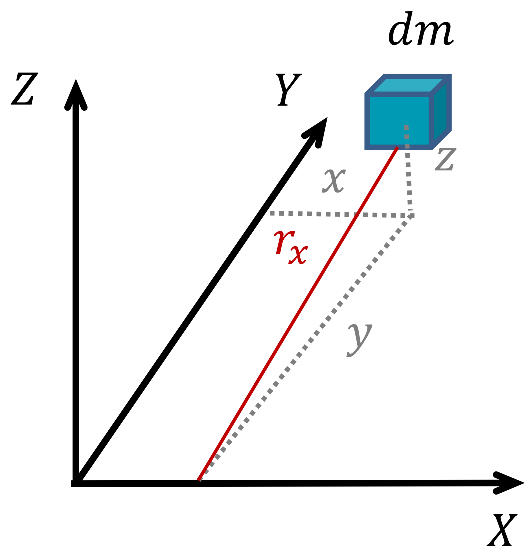
Mass Moment of Inertia

$$I = \int_0^M r^2 dm$$

$$I_x = \int r_x^2 \, dm = \int (y^2 + z^2) \, dm$$

$$I_y = \int r_y^2 \, dm = \int (z^2 + x^2) \, dm$$

$$I_z = \int r_z^2 \, dm = \int (x^2 + y^2) \, dm$$



x is the distance from the YZ-plane to dm.
y is the distance from the ZX-plane to dm.
z is the distance from the XY-plane to dm.

Mass Moment of Inertia

It has several names:

1- Moment of inertia.
2- Second moment of mass.
3- Angular mass.
4- Rotational inertia.

It should not be confused with the second moment of area!

Mass Moment of Inertia

Second moment of area:

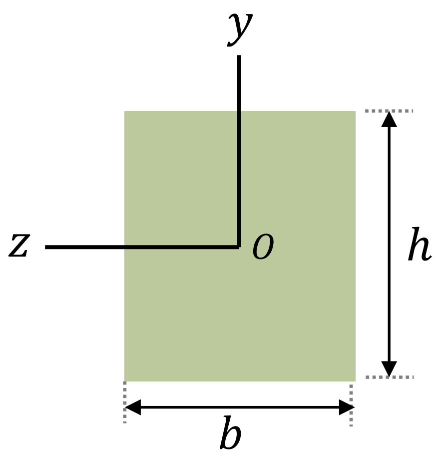
(the area moment of inertia):

$$I_Z = \int_A y^2 \, dA$$

For a rectangular cross section: $I_Z = \frac{bh^3}{12}$

For a circular cross section: $I_Z = \frac{\pi}{64} d^4$

$$\kappa = \frac{M}{EI_z}$$



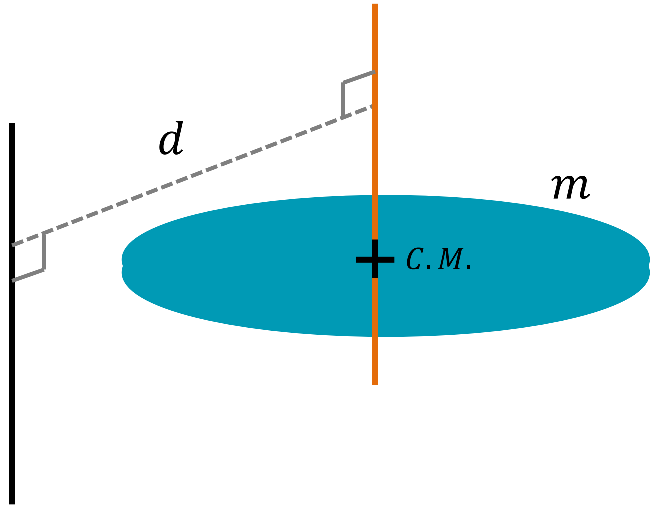
Mass Moment of Inertia

Mass moment of inertia	Second moment of area
$$I_z = \int_0^M (y^2 + z^2) dm$$	$$I_z = \int_A y^2 \, dA$$
Resistance to change in its rotational speed about a specific axis	Resistance to bending about a specific axis
A measure of the distribution of the mass of an object relative to a given axis.	A measure of the distribution of the area of a cross-section relative to a given axis.
Units: $kg.m^2$	Units: m^4

Mass Moment of Inertia

Parallel-axis theorem:

The moment of inertia about any axis can be calculated from the moment of inertia about a parallel axis that passes through the center of mass.



$$I = I_G + md^2$$

where:

d : is the distance between the two axes.

m : is the total mass of the body.

Mass Moment of Inertia

For an object rotating about a certain axis with an angular speed of (ω), the angular momentum can be found from:

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where

$I_{xx} = \int (y^2 + z^2)\, dm$: moment of inertia about the x-axis

$I_{yy} = \int (z^2 + x^2)\, dm$: moment of inertia about the y-axis

$I_{zz} = \int (x^2 + y^2)\, dm$: moment of inertia about the z-axis

$I_{xy} = I_{yx} = -\int xy\, dm$: xy product of inertia

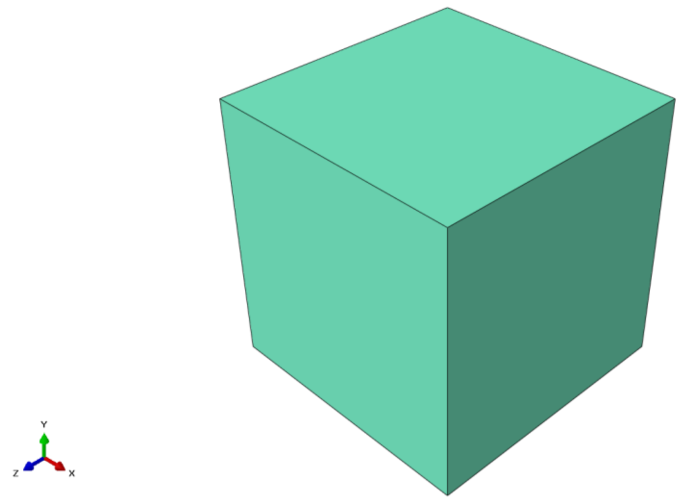
$I_{yz} = I_{zy} = -\int yz\, dm$: yz product of inertia

$I_{xz} = I_{zx} = -\int xz\, dm$: xz product of inertia

Mass Moment of Inertia

Example:

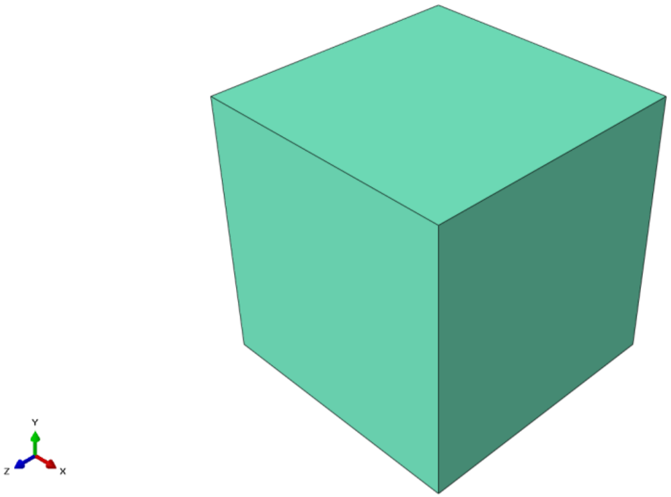
A cube of steel (density of $7700\,\frac{kg}{m^3}$) with an edge of length of 0.25 m.



$$I = \frac{1}{12} m(h^2 + w^2)$$

$$I = \frac{1}{12}\left(7700\,\frac{kg}{m^3} \times (0.25^3)\right)(0.25^2 + 0.25^2) = 1.25325\ \mathrm{kg.\,m^2}$$

Mass Moment of Inertia

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Mass properties:
  Volume: 0.0156
  Volume centroid: -19.88,-19.88,0.125
  Mass: 120.31
  Center of mass: -19.88,-19.88,0.125
  Moment of inertia about the center of mass (Ixx, Iyy, Izz, Ixy, Iyz, Izx): 1.25,1.25,1.25,0,0,0
```