Distance Learning Initiative

Introduction to Robotics

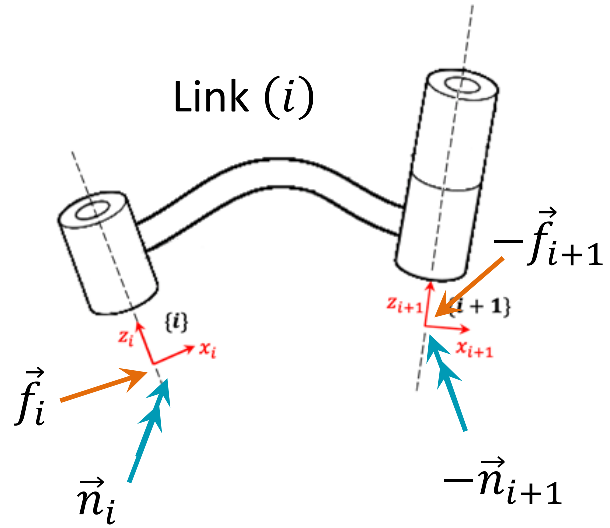
Static Forces and Torques

Dr. Firas Jarrar

2020

Static Forces and Torques

Question: How to find the joint forces or torques required to keep the robotic arm in static equilibrium?



$[f_i]$: the force acting on joint (i) by link $(i-1)$

$[f_{i+1}]$: the force acting on joint $(i+1)$ by link (i).

$[n_i]$: the moment acting on joint (i) by link $(i-1)$

$[n_{i+1}]$: the moment acting on joint $(i+1)$ by link (i).

Static Forces and Torques

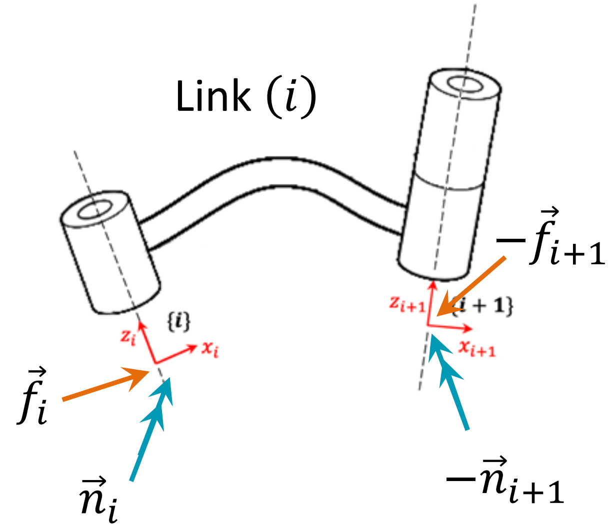
Static Forces:

$$\sum Forces = 0$$

$$[\,{}^{i}f_i] - [\,{}^{i}f_{i+1}] = 0$$

$$[\,{}^{i}f_i] - [{}_{i+1}^{\;\;i}R][{}^{i+1}f_{i+1}] = 0$$

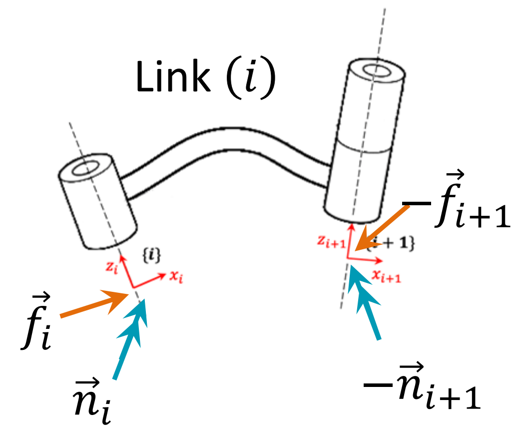
$$[\,{}^{i}f_i] = [{}_{i+1}^{\;\;i}R][{}^{i+1}f_{i+1}]$$



Note: The weight of the link is neglected.

Static Forces and Torques

Static Moments:



$$\sum moments = 0$$

$$[\ {}^{i}n_i] - [\ {}^{i}n_{i+1}] + [{}^{i}P_{i+1}\ ] \times (-[\ {}^{i}f_{i+1}]) = 0$$

$$[\ {}^{i}n_i] - [\ {}^{i}n_{i+1}] - [{}^{i}P_{i+1}\ ] \times [\ {}^{i}f_{i+1}] = 0$$

$$[\ {}^{i}n_i] - [{}_{i+1}^{\ \ i}R][{}^{i+1}n_{i+1}] - [{}^{i}P_{i+1}\ ] \times [{}_{i+1}^{\ \ i}R][{}^{i+1}f_{i+1}] = 0$$

$$[\ {}^{i}n_i] = [{}_{i+1}^{\ \ i}R][{}^{i+1}n_{i+1}] + [{}^{i}P_{i+1}\ ] \times [{}_{i+1}^{\ \ i}R][{}^{i+1}f_{i+1}]$$

$$\boxed{[\ {}^{i}n_i] = [{}_{i+1}^{\ \ i}R][{}^{i+1}n_{i+1}] + [{}^{i}P_{i+1}\ ] \times [\ {}^{i}f_i]}$$

Static Forces and Torques

$$[\,{}^{i}f_i] = [{}^{i}_{i+1}R][{}^{i+1}f_{i+1}]$$

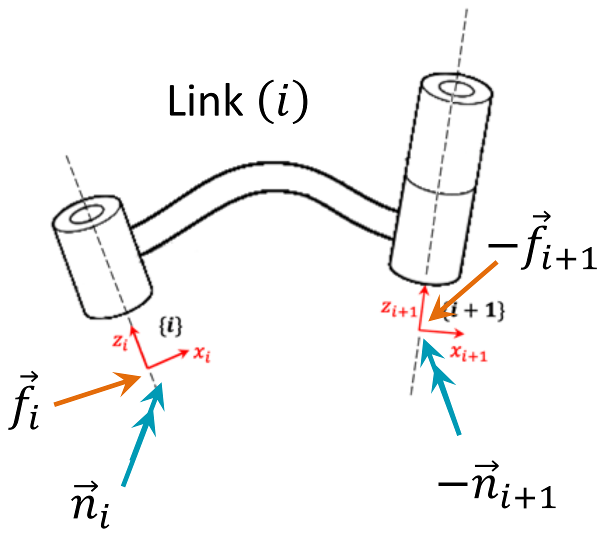
$$[\,{}^{i}n_i] = [{}^{i}_{i+1}R][{}^{i+1}n_{i+1}] + [{}^{i}P_{i+1}] \times [\,{}^{i}f_i]$$

For a revolute joint, the actuator torque is:

$$\tau_i = -[\,{}^{i}n_i].\ {}^{i}\begin{bmatrix}0\\0\\1\end{bmatrix} = -[\,{}^{i}n_i]^T\ {}^{i}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

For a prismatic joint, the actuator force is:

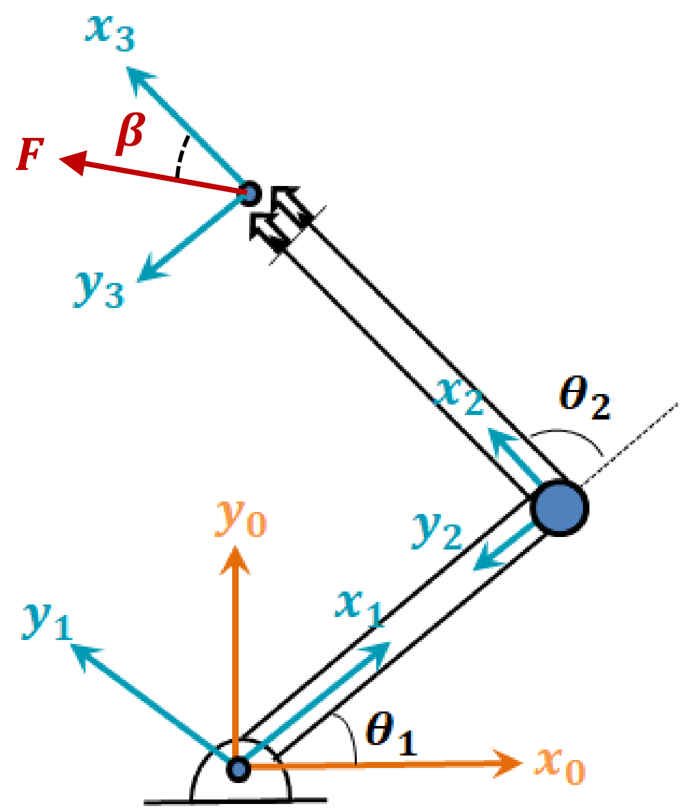
$$\tau_i = -[\,{}^{i}f_i].\ {}^{i}\begin{bmatrix}0\\0\\1\end{bmatrix} = -[\,{}^{i}f_i]^T\ {}^{i}\begin{bmatrix}0\\0\\1\end{bmatrix}$$



Static Forces and Torques

Example:

For the planar 2 DOF RR robotic arm, calculate the static torques required for each joint motor? Assume that the end-effector is subjected to a planar force vector, $\vec{F}$.



Static Forces and Torques

For $i = 2$:

$$[\,{}^{2}f_2] = [{}^{2}_{3}R][\,{}^{3}f_3]$$

$$[\,{}^{2}f_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Fc_\beta \\ Fs_\beta \\ 0 \end{bmatrix}$$

$$[\,{}^{2}f_2] = \begin{bmatrix} Fc_\beta \\ Fs_\beta \\ 0 \end{bmatrix}$$

$$[\,{}^{i}f_i] = [{}^{i}_{i+1}R][{}^{i+1}f_{i+1}]$$

$$[\,{}^{i}n_i] = [{}^{i}_{i+1}R][{}^{i+1}n_{i+1}] + [{}^{i}P_{i+1}] \times [\,{}^{i}f_i]$$

For a revolute joint: $\tau_i = -[\,{}^{i}n_i]^T \; {}^{i}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

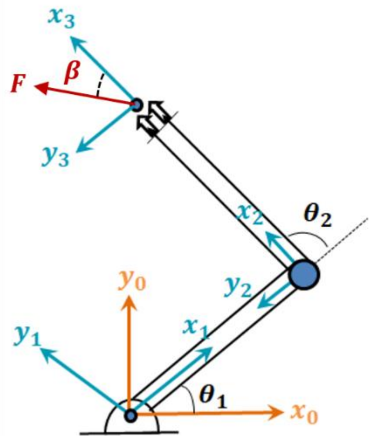
For a prismatic joint: $\tau_i = -[\,{}^{i}f_i]^T \; {}^{i}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$[S(a)] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$[{}^{0}_{1}R] = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^{1}_{2}R] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^{2}_{3}R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[{}^{0}P_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; ; \; [{}^{1}P_2] = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \; ; \; [{}^{2}P_3] = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$$

Static Forces and Torques

$$[\,^2f_2] = \begin{bmatrix} Fc_\beta \\ Fs_\beta \\ 0 \end{bmatrix}$$

For $i = 2$:

$$[\,^2n_2] = [^2_3R][\,^3n_3] + [^2P_3] \times [\,^2f_2]$$

$$[\,^2n_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} Fc_\beta \\ Fs_\beta \\ 0 \end{bmatrix}$$

$$[\,^2n_2] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_2 \\ 0 & L_2 & 0 \end{bmatrix}\begin{bmatrix} Fc_\beta \\ Fs_\beta \\ 0 \end{bmatrix}$$

$$[\,^2n_2] = \begin{bmatrix} 0 \\ 0 \\ L_2Fs_\beta \end{bmatrix}$$

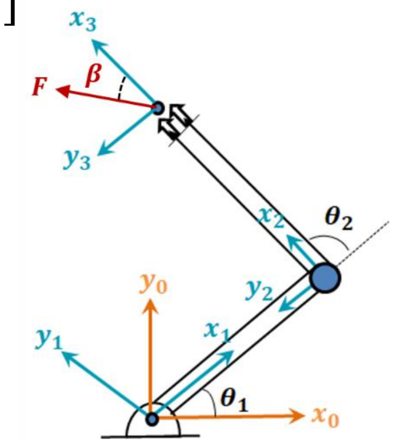
$$[\,^if_i] = [^i_{i+1}R][^{i+1}f_{i+1}]$$
$$[\,^in_i] = [^i_{i+1}R][^{i+1}n_{i+1}] + [^iP_{i+1}] \times [\,^if_i]$$

For a revolute joint: $\tau_i = -[\,^in_i]^T \; {}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For a prismatic joint: $\tau_i = -[\,^if_i]^T \; {}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$[S(a)] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$[^0_1R] = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$[^1_2R] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$[^2_3R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[^0P_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; ; \; [^1P_2] = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \; ; \; [^2P_3] = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$$

Static Forces and Torques

$$[\,^2f_2] = \begin{bmatrix} Fc_\beta \\ Fs_\beta \\ 0 \end{bmatrix} \; , \; [\,^2n_2] = \begin{bmatrix} 0 \\ 0 \\ L_2Fs_\beta \end{bmatrix}$$

For $i = 2$:

$$\tau_2 = -[\,^2n_2]^T \; {}^2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\tau_2 = -[0 \quad 0 \quad L_2Fs_\beta]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -L_2Fs_\beta$$

$$[\,^if_i] = [\,^i_{i+1}R][\,^{i+1}f_{i+1}]$$

$$[\,^in_i] = [\,^i_{i+1}R][\,^{i+1}n_{i+1}] + [\,^iP_{i+1}] \times [\,^if_i]$$

For a revolute joint: $\tau_i = -[\,^in_i]^T \; {}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

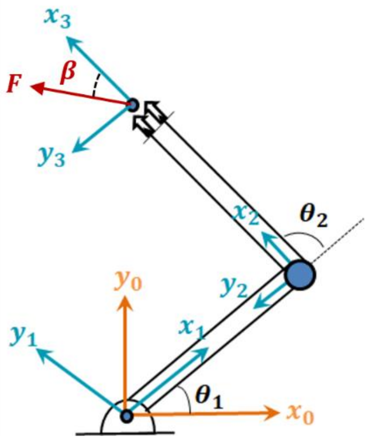
For a prismatic joint: $\tau_i = -[\,^if_i]^T \; {}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$[S(a)] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$[\,^0_1R] = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\,^1_2R] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\,^2_3R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[\,^0P_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; ; \; [\,^1P_2] = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \; ; \; [\,^2P_3] = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$$

Static Forces and Torques

$$[\,{}^2f_2] = \begin{bmatrix} Fc_\beta \\ Fs_\beta \\ 0 \end{bmatrix}, [\,{}^2n_2] = \begin{bmatrix} 0 \\ 0 \\ L_2Fs_\beta \end{bmatrix}, \tau_2 = -L_2Fs_\beta$$

For $i = 1$:

$$[\,{}^1f_1] = [{}^1_2R][\,{}^2f_2]$$

$$[\,{}^1f_1] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Fc_\beta \\ Fs_\beta \\ 0 \end{bmatrix}$$

$$[\,{}^1f_1] = \begin{bmatrix} Fc_\beta c_2 - Fs_\beta s_2 \\ Fc_\beta s_2 + Fs_\beta c_2 \\ 0 \end{bmatrix}$$

$$[\,{}^1f_1] = \begin{bmatrix} F\cos(\beta + \theta_2) \\ F\sin(\beta + \theta_2) \\ 0 \end{bmatrix}$$

$$[\,{}^if_i] = [{}^i_{i+1}R][{}^{i+1}f_{i+1}]$$

$$[\,{}^in_i] = [{}^i_{i+1}R][{}^{i+1}n_{i+1}] + [{}^iP_{i+1}] \times [\,{}^if_i]$$

For a revolute joint: $\tau_i = -[\,{}^in_i]^T \; {}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

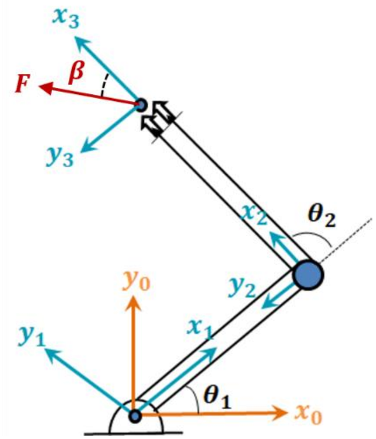
For a prismatic joint: $\tau_i = -[\,{}^if_i]^T \; {}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$[S(a)] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$[{}^0_1R] = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^1_2R] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^2_3R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[{}^0P_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; ; \; [{}^1P_2] = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \; ; \; [{}^2P_3] = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$$

Static Forces and Torques

$$[\,{}^2f_2] = \begin{bmatrix} Fc_\beta \\ Fs_\beta \\ 0 \end{bmatrix}, [\,{}^2n_2] = \begin{bmatrix} 0 \\ 0 \\ L_2Fs_\beta \end{bmatrix}, \qquad \tau_2 = -L_2Fs_\beta$$

$$[\,{}^1f_1] = \begin{bmatrix} F\cos(\beta+\theta_2) \\ F\sin(\beta+\theta_2) \\ 0 \end{bmatrix}$$

For $i = 1$:

$$[\,{}^1n_1] = [{}^1_2R][\,{}^2n_2] + [{}^1P_2] \times [\,{}^1f_1]$$

$$[\,{}^1n_1] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ L_2Fs_\beta \end{bmatrix} + \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} F\cos(\beta+\theta_2) \\ F\sin(\beta+\theta_2) \\ 0 \end{bmatrix}$$

$$[\,{}^1n_1] = \begin{bmatrix} 0 \\ 0 \\ L_2Fs_\beta \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_1 \\ 0 & L_1 & 0 \end{bmatrix} \begin{bmatrix} F\cos(\beta+\theta_2) \\ F\sin(\beta+\theta_2) \\ 0 \end{bmatrix}$$

$$[\,{}^1n_1] = \begin{bmatrix} 0 \\ 0 \\ L_2Fs_\beta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_1F\sin(\beta+\theta_2) \end{bmatrix}$$

$$[\,{}^1n_1] = \begin{bmatrix} 0 \\ 0 \\ L_2Fs_\beta + L_1F\sin(\beta+\theta_2) \end{bmatrix}$$

$$[\,{}^if_i] = [{}^i_{i+1}R][{}^{i+1}f_{i+1}]$$

$$[\,{}^in_i] = [{}^i_{i+1}R][{}^{i+1}n_{i+1}] + [{}^iP_{i+1}] \times [\,{}^if_i]$$

For a revolute joint: $\tau_i = -[\,{}^in_i]^T \; {}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

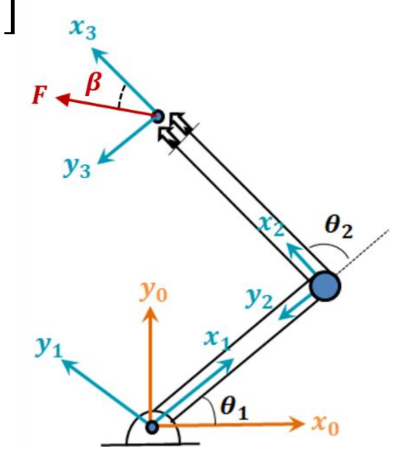
For a prismatic joint: $\tau_i = -[\,{}^if_i]^T \; {}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$[S(a)] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$[{}^0_1R] = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^1_2R] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^2_3R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[{}^0P_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \;;\; [{}^1P_2] = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \;;\; [{}^2P_3] = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$$

Static Forces and Torques

$$[\,^2f_2] = \begin{bmatrix} F c_\beta \\ F s_\beta \\ 0 \end{bmatrix}, [\,^2n_2] = \begin{bmatrix} 0 \\ 0 \\ L_2 F s_\beta \end{bmatrix}, \qquad \tau_2 = -L_2 F s_\beta$$

$$[\,^1f_1] = \begin{bmatrix} F\cos(\beta+\theta_2) \\ F\sin(\beta+\theta_2) \\ 0 \end{bmatrix}, [\,^1n_1] = \begin{bmatrix} 0 \\ 0 \\ L_2 F s_\beta + L_1 F \sin(\beta+\theta_2) \end{bmatrix}$$

For $i = 2$:

$$\tau_1 = -[\,^1n_1]^T \;{}^2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\tau_1 = -[0 \quad 0 \quad L_2 F s_\beta + L_1 F \sin(\beta+\theta_2)] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\tau_1 = -L_2 F s_\beta - L_1 F \sin(\beta+\theta_2)$$

$$[\,^if_i] = [\,_{i+1}^{\;\;i}R][\,^{i+1}f_{i+1}]$$

$$[\,^in_i] = [\,_{i+1}^{\;\;i}R][\,^{i+1}n_{i+1}] + [\,^iP_{i+1}] \times [\,^if_i]$$

For a revolute joint: $\tau_i = -[\,^in_i]^T \;{}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

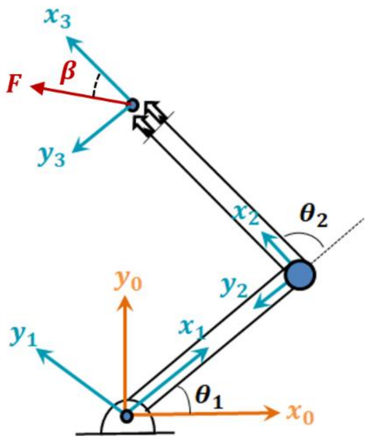
For a prismatic joint: $\tau_i = -[\,^if_i]^T \;{}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$[S(a)] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$[\,_1^0R] = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\,_2^1R] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\,_3^2R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[\,^0P_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; ; \; [\,^1P_2] = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \; ; \; [\,^2P_3] = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$$

Static Forces and Torques

$$[\,{}^2f_2] = \begin{bmatrix} Fc_\beta \\ Fs_\beta \\ 0 \end{bmatrix}, [\,{}^2n_2] = \begin{bmatrix} 0 \\ 0 \\ L_2Fs_\beta \end{bmatrix}, \qquad \tau_2 = -L_2Fs_\beta$$

$$[\,{}^1f_1] = \begin{bmatrix} F\cos(\beta+\theta_2) \\ F\sin(\beta+\theta_2) \\ 0 \end{bmatrix}, [\,{}^1n_1] = \begin{bmatrix} 0 \\ 0 \\ L_2Fs_\beta + L_1F\sin(\beta+\theta_2) \end{bmatrix}$$

$$\tau_1 = -L_2Fs_\beta - L_1F\sin(\beta+\theta_2)$$

$$\tau_1 = -L_2Fs_\beta - L_1F\sin(\beta+\theta_2)$$

$$\tau_1 = -L_2Fs_\beta - L_1Fs_\beta c_2 - L_1Fc_\beta s_2$$

$$\tau_1 = -(L_1s_2)Fc_\beta - (L_1c_2 + L_2)Fs_\beta$$

$$\tau_2 = -L_2Fs_\beta$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = -\begin{bmatrix} L_1s_2 & (L_1c_2+L_2) \\ 0 & L_2 \end{bmatrix}\begin{bmatrix} Fc_\beta \\ Fs_\beta \end{bmatrix}$$

$$[\,{}^if_i] = [{}^i_{i+1}R][{}^{i+1}f_{i+1}]$$
$$[\,{}^in_i] = [{}^i_{i+1}R][{}^{i+1}n_{i+1}] + [{}^iP_{i+1}] \times [\,{}^if_i]$$

For a revolute joint: $\tau_i = -[\,{}^in_i]^T \; {}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

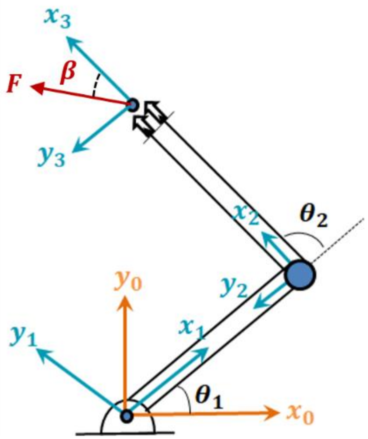
For a prismatic joint: $\tau_i = -[\,{}^if_i]^T \; {}^i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$[S(a)] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$[{}^0_1R] = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^1_2R] = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[{}^2_3R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[{}^0P_1] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; ; \; [{}^1P_2] = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \; ; \; [{}^2P_3] = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$$

Static Forces and Torques

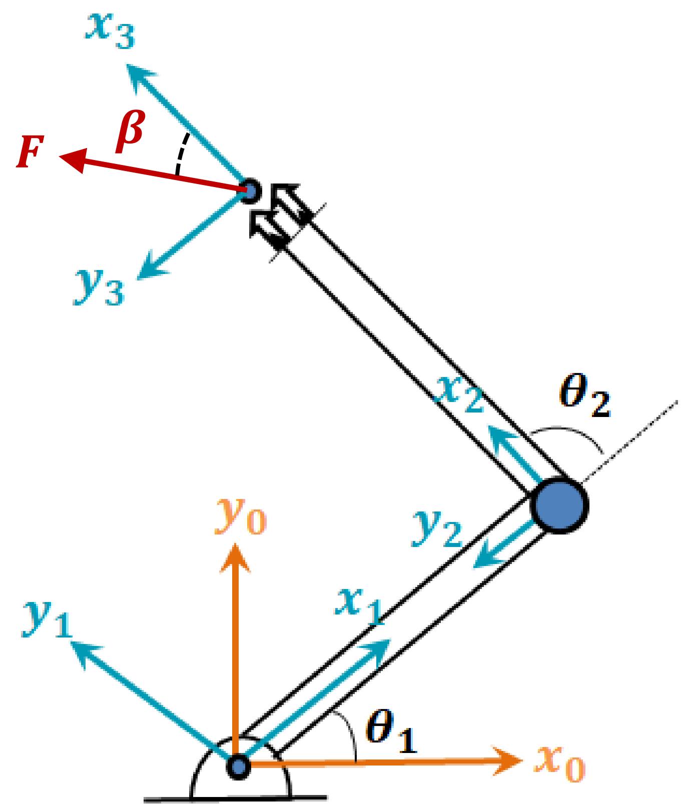
$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = -\begin{bmatrix}L_1 s_2 & (L_1 c_2 + L_2)\\ 0 & L_2\end{bmatrix}\begin{bmatrix}F c_\beta\\ F s_\beta\end{bmatrix}$$

Compare to

$${}^3\begin{bmatrix}\dot{x}_3\\ \dot{y}_3\end{bmatrix} = \begin{bmatrix}L_1 s_2 & 0\\ L_1 c_2 + L_2 & L_2\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$[\tau] = -\,{}^3[\mathrm{J}_v]^T[\,{}^3 f_3]$$

$$[\tau] = -{}^{n+1}[\mathrm{J}_v]^T[{}^{n+1} f_{n+1}]$$



Static Forces and Torques

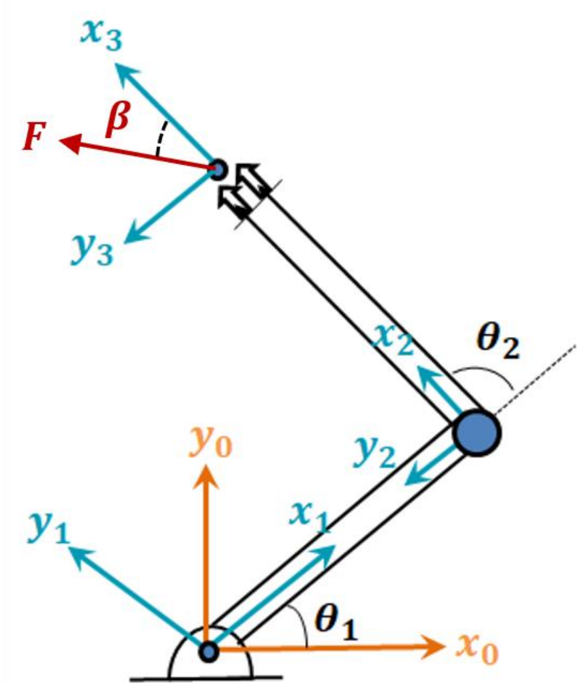
$$^{0}[J_v]^T[\,^{0}f_3] = ?\ ;\quad [\,^{0}f_3] = [^{0}_{3}R][\,^{3}f_3]$$

$$^{0}\begin{bmatrix}\dot{x}_3 \\ \dot{y}_3\end{bmatrix} = \begin{bmatrix}-(L_1 s_1 + L_2 s_{12}) & -L_2 s_{12} \\ (L_1 c_1 + L_2 c_{12}) & L_2 c_{12}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}$$

$$^{0}[J_v] = \begin{bmatrix}-(L_1 s_1 + L_2 s_{12}) & -L_2 s_{12} \\ (L_1 c_1 + L_2 c_{12}) & L_2 c_{12}\end{bmatrix}$$

$$[\,^{0}f_3] = {}^{0}\begin{bmatrix}F\cos(\beta + \theta_1 + \theta_2) \\ F\sin(\beta + \theta_1 + \theta_2)\end{bmatrix}$$

$$^{0}[J_v]^T[\,^{0}f_3] = \begin{bmatrix}-(L_1 s_1 + L_2 s_{12}) & (L_1 c_1 + L_2 c_{12}) \\ -L_2 s_{12} & L_2 c_{12}\end{bmatrix}\begin{bmatrix}F\cos(\beta + \theta_1 + \theta_2) \\ F\sin(\beta + \theta_1 + \theta_2)\end{bmatrix}$$



Static Forces and Torques

$$^{0}[J_v]^T[\,^{0}f_3] = \begin{bmatrix} -(L_1s_1+L_2s_{12}) & (L_1c_1+L_2c_{12}) \\ -L_2s_{12} & L_2c_{12} \end{bmatrix} \begin{bmatrix} F\cos(\beta+\theta_1+\theta_2) \\ F\sin(\beta+\theta_1+\theta_2) \end{bmatrix}$$

$$^{0}[J_v]^T[\,^{0}f_3] = \begin{bmatrix} -(L_1s_1+L_2s_{12})F\cos(\beta+\theta_1+\theta_2)+(L_1c_1+L_2c_{12})F\sin(\beta+\theta_1+\theta_2) \\ -L_2s_{12}F\cos(\beta+\theta_1+\theta_2)+L_2c_{12}F\sin(\beta+\theta_1+\theta_2) \end{bmatrix}$$

$$^{0}[J_v]^T[\,^{0}f_3] = F\begin{bmatrix} -(L_1s_1+L_2s_{12})\left(c_\beta c_{12}-s_\beta s_{12}\right)+(L_1c_1+L_2c_{12})\left(s_\beta c_{12}+c_\beta s_{12}\right) \\ -L_2s_{12}\left(c_\beta c_{12}-s_\beta s_{12}\right)+L_2c_{12}\left(s_\beta c_{12}+c_\beta s_{12}\right) \end{bmatrix}$$

$$^{0}[J_v]^T[\,^{0}f_3] = F\begin{bmatrix} -(L_1s_1+L_2s_{12})c_\beta c_{12}+(L_1s_1+L_2s_{12})s_\beta s_{12}+(L_1c_1+L_2c_{12})s_\beta c_{12}+(L_1c_1+L_2c_{12})c_\beta s_{12} \\ -L_2s_{12}c_\beta c_{12}+L_2s_{12}s_\beta s_{12}+L_2c_{12}s_\beta c_{12}+L_2c_{12}c_\beta s_{12} \end{bmatrix}$$

$$^{0}[J_v]^T[\,^{0}f_3] = F\begin{bmatrix} (-(L_1s_1+L_2s_{12})c_{12}+(L_1c_1+L_2c_{12})s_{12})c_\beta+\left((L_1s_1+L_2s_{12})s_{12}+(L_1c_1+L_2c_{12})c_{12}\right)s_\beta \\ (-L_2s_{12}c_{12}+L_2c_{12}s_{12})c_\beta+(L_2s_{12}s_{12}+L_2c_{12}c_{12})s_\beta \end{bmatrix}$$

$$^{0}[J_v]^T[\,^{0}f_3] = F\begin{bmatrix} (-L_1s_1c_{12}+L_1c_1s_{12}-L_2s_{12}c_{12}+L_2c_{12}s_{12})c_\beta+(L_1s_1s_{12}+L_1c_1c_{12}+L_2s_{12}s_{12}+L_2c_{12}c_{12})s_\beta \\ (-L_2s_{12}c_{12}+L_2c_{12}s_{12})c_\beta+(L_2s_{12}s_{12}+L_2c_{12}c_{12})s_\beta \end{bmatrix}$$

$$^{0}[J_v]^T[\,^{0}f_3] = F\begin{bmatrix} (L_1s_2)c_\beta+(L_1c_2+L_2)s_\beta \\ (0)c_\beta+(L_2)s_\beta \end{bmatrix}$$

$$^{0}[J_v]^T[\,^{0}f_3] = \begin{bmatrix} L_1s_2Fc_\beta+(L_1c_2+L_2)Fs_\beta \\ L_2Fs_\beta \end{bmatrix} = \begin{bmatrix} L_1s_2 & (L_1c_2+L_2) \\ 0 & L_2 \end{bmatrix}\begin{bmatrix} Fc_\beta \\ Fs_\beta \end{bmatrix} = \,^{3}[J_v]^T[\,^{3}f_3]$$

Static Forces and Torques

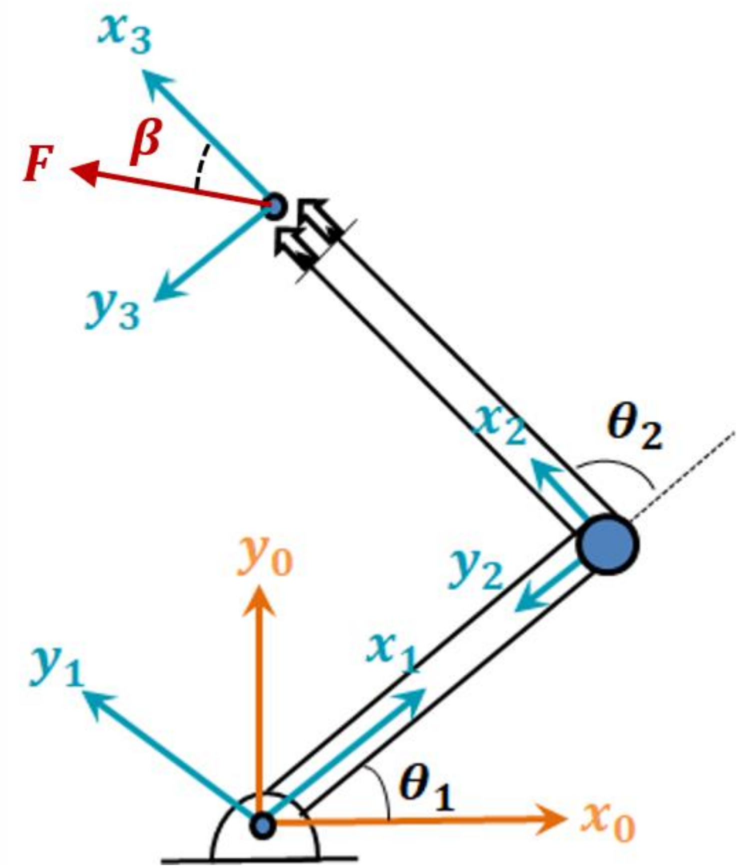
Which one is better to use:

$$[\tau] = -{}^{n+1}[\mathrm{J}_v]^T\left[{}^{n+1}f_{n+1}\right]$$

Or

$$[\tau] = -\,{}^{0}[\mathrm{J}_v]^T\left[\,{}^{0}f_{n+1}\right]$$

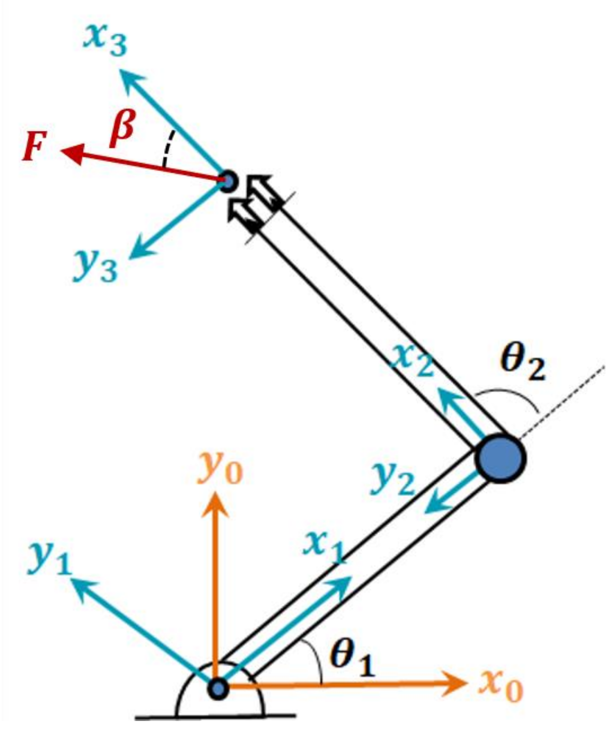
??



Static Forces and Torques

If $\vec{F}$ is constant in frame {0}, such as in the case of the weight of the payload, then use $[\tau] = -\,{}^{0}[\mathrm{J}_v]^T[\,{}^{0}f_{n+1}]$.

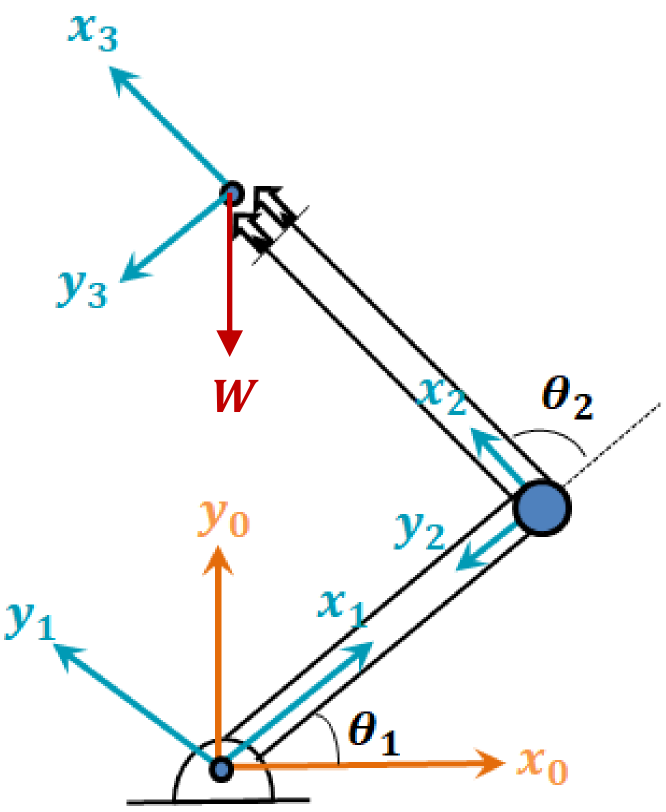
If $\vec{F}$ is constant in frame {n+1}, then use $[\tau] = -{}^{n+1}[\mathrm{J}_v]^T[{}^{n+1}f_{n+1}]$.



Static Forces and Torques

Example:

For the planar 2 DOF RR robotic arm, calculate the static torques required for each joint motor? Assume that the weight of the payload is W.



Static Forces and Torques

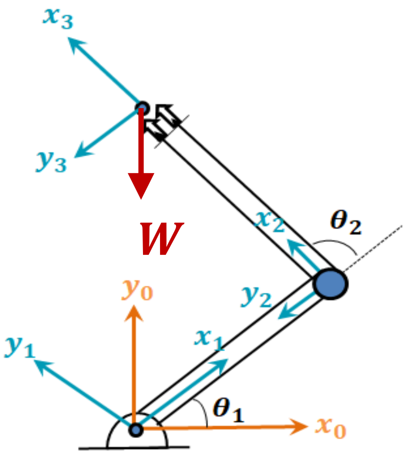
$$[\tau] = -\,{}^{0}[\mathrm{J}_v]^T [\,{}^{0}f_3]$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = -\begin{bmatrix}-(L_1s_1+L_2s_{12}) & -L_2s_{12}\\ (L_1c_1+L_2c_{12}) & L_2c_{12}\end{bmatrix}^T \begin{bmatrix}0\\ -W\end{bmatrix}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = -\begin{bmatrix}-(L_1s_1+L_2s_{12}) & (L_1c_1+L_2c_{12})\\ -L_2s_{12} & L_2c_{12}\end{bmatrix} \begin{bmatrix}0\\ -W\end{bmatrix}$$

$$\tau_1 = (L_1c_1+L_2c_{12})W$$

$$\tau_2 = L_2c_{12}W$$



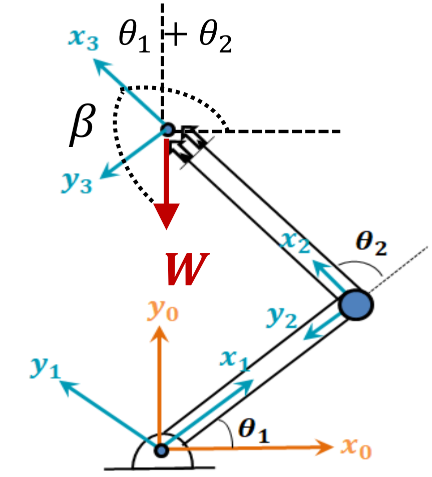
Static Forces and Torques

$$[\tau] = -\,^{3}[J_v]^T[\,^{3}f_3]$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = -\begin{bmatrix}L_1 s_2 & 0\\ L_1 c_2 + L_2 & L_2\end{bmatrix}^T \begin{bmatrix}F c_\beta\\ F s_\beta\end{bmatrix}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = -\begin{bmatrix}L_1 s_2 & (L_1 c_2 + L_2)\\ 0 & L_2\end{bmatrix}\begin{bmatrix}-W\sin(\theta_1+\theta_2)\\ -W\cos(\theta_1+\theta_2)\end{bmatrix}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = -\begin{bmatrix}L_1 s_2 & (L_1 c_2 + L_2)\\ 0 & L_2\end{bmatrix}\begin{bmatrix}-W s_{12}\\ -W c_{12}\end{bmatrix}$$



$$\beta = 270 - (\theta_1 + \theta_2)$$

$$\tau_1 = L_1 s_2 W s_{12} + (L_1 c_2 + L_2) W c_{12}$$
$$\tau_1 = (L_1 s_2 s_{12} + L_1 c_2 c_{12} + L_2 c_{12})W$$
$$\tau_1 = (L_1(s_2 s_{12} + c_2 c_{12}) + L_2 c_{12})W = (L_1 c_1 + L_2 c_{12})W$$
$$\tau_2 = L_2 c_{12} W$$

Static Forces and Torques

$$\tau_1 = (L_1 c_1 + L_2 c_{12})W$$

$$\tau_2 = L_2 c_{12} W$$

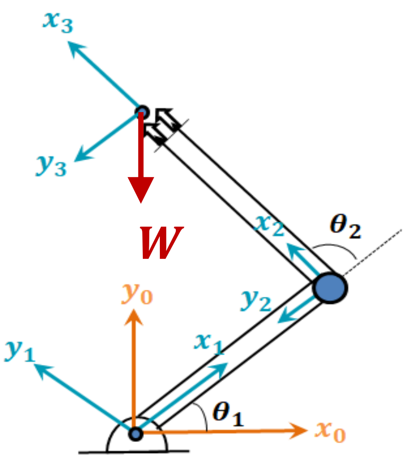
The cosine has a maximum value of 1 when $\theta = 0$

When both $\theta_1 = 0$ and $\theta_2 = 0$ (fully extended and horizonal),

$$\tau_{1,max} = (L_1 + L_2)W$$

$$\tau_{2,max} = (L_2)W$$

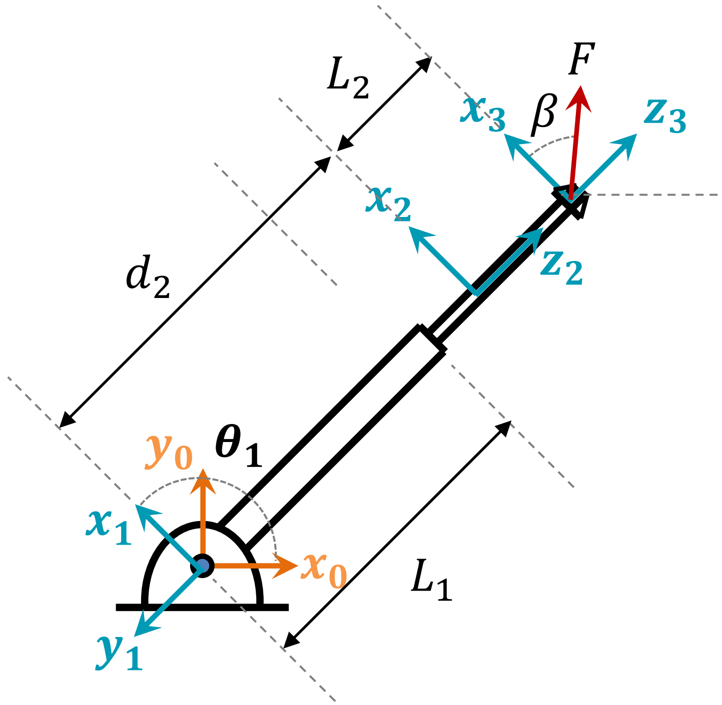
These can be used for motor selection.



Static Forces and Torques

Example:

For the planar 2 DOF RP robotic arm, calculate the static torque required for joint (1) and the static force required for joint (2)?



Static Forces and Torques

$$[\tau] = -\,{}^{3}[\mathrm{J}_v]^T[\,{}^{3}f_3]$$

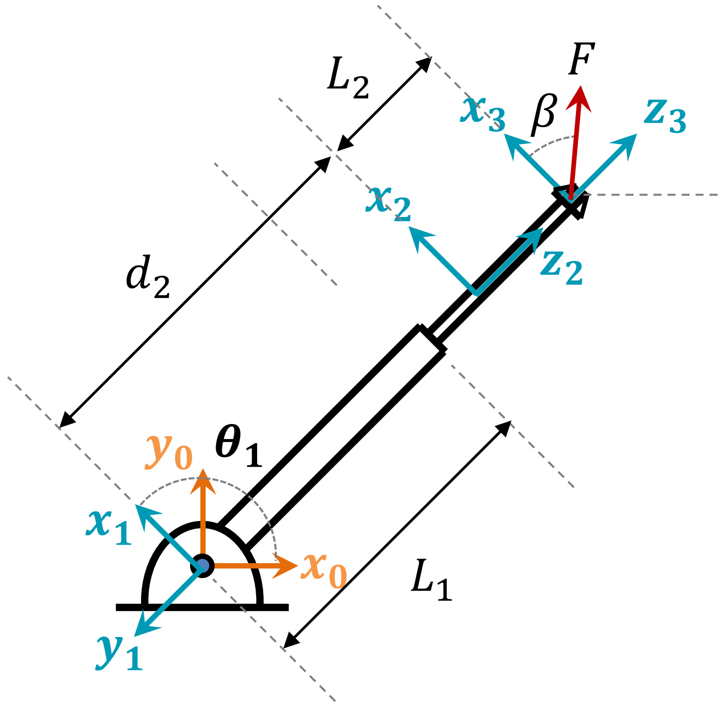
$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = -\begin{bmatrix}(d_2+L_2) & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}^T \begin{bmatrix}Fc_\beta\\ 0\\ Fs_\beta\end{bmatrix}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = -\begin{bmatrix}(d_2+L_2) & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \begin{bmatrix}Fc_\beta\\ 0\\ Fs_\beta\end{bmatrix}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = -\begin{bmatrix}(d_2+L_2)Fc_\beta\\ Fs_\beta\end{bmatrix}$$

$$\tau_1 = -(d_2+L_2)Fc_\beta$$

$$\tau_2 = -Fs_\beta$$



Static Forces and Torques

For more details on this subject, please see:

- Introduction to Robotics: Mechanics and Control, by John J. Craig, 3rd Edition, Addison-Wesley Publishing Company, 2003.