Distance Learning Initiative

Applied Finite Element Analysis

Matrix Algebra

Dr. Firas Jarrar

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Definition of a Matrix

Matrix:

A matrix is an ordered set of numbers listed in a rectangular form.

For example:

$$\begin{bmatrix} 8 & 0 & 17 & -2 \\ 1.3 & 5 & -9 & 0.1 \end{bmatrix}$$

This matrix has an order of 2 x 4.

“Wake up, Neo.
The Matrix has you.”

The Matrix 1999

Row Matrix

Row Matrix:

This is a matrix with only one row. For example:

$$\begin{bmatrix} 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 9 \end{bmatrix}$$

Column Matrix

Column Matrix:

This is a matrix with only one column. For example:

$$\begin{bmatrix} 2 \\ 0.9 \end{bmatrix} \quad , \quad \begin{bmatrix} -4 \\ 1 \\ 7 \\ 7 \end{bmatrix}$$

A Square Matrix:

A square matrix has the same number of rows and columns. For example:

$$\begin{bmatrix} 9 & 1 \\ -5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8 & 7 \\ -2 & 1 & -0.3 \\ 10 & -0.2 & 4 \end{bmatrix}$$

Diagonal Elements

Diagonal Elements:

In a square matrix the elements $a_{i,i}$, with i = 1,2,3,... , are called diagonal elements:

$$\begin{bmatrix} 9 & 1 \\ -5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8 & 7 \\ -2 & 1 & -0.3 \\ 10 & -0.2 & 4 \end{bmatrix}$$

Diagonal Matrix

Diagonal Matrix:

This is a square matrix with all the non-diagonal elements equal to zero. For example:

$$\begin{bmatrix} 9 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Identity Matrix:

This is a diagonal matrix with all of the diagonal elements equal to 1. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Opposite of a Matrix:

Matrix [B] is said to be the opposite of [A], if $b_{i,j} = -a_{i,j}$ for all elements in [B]. For example:

$$[A] = \begin{bmatrix} 7 & 8 & 7 \\ -2 & 1 & -0.3 \\ 10 & -0.2 & 4 \end{bmatrix}$$

$$[B] = \begin{bmatrix} -7 & -8 & -7 \\ 2 & -1 & 0.3 \\ -10 & 0.2 & -4 \end{bmatrix}$$

Transpose of a Matrix:

Matrix [B] is said to be the transpose of [A], if each column in [B] is equal to the corresponding row in [A]. For example:

$$[A] = \begin{bmatrix} 7 & 8 & 7 \\ -2 & 1 & -0.3 \\ 10 & -0.2 & 4 \end{bmatrix}$$

$$[B] = [A]^T = \begin{bmatrix} 7 & -2 & 10 \\ 8 & 1 & -0.2 \\ 7 & -0.3 & 4 \end{bmatrix}$$

Transpose of a Matrix

Example:

$$[A] = \begin{bmatrix} 1 & 0 & -3 \\ 7 & 8 & 3 \end{bmatrix}$$

$$[B] = [A]^T = \begin{bmatrix} 1 & 7 \\ 0 & 8 \\ -3 & 3 \end{bmatrix}$$

Note that: $([A][B])^T=[B]^T[A]^T$

$([A][B][C])^T=[C]^T[B]^T[A]^T$

Symmetric Matrix

Symmetric Matrix:

A square matrix is called symmetric, if it is equal to its transpose.

For example:

$$\begin{bmatrix} 7 & 8 & 7 \\ 8 & 1 & -0.2 \\ 7 & -0.2 & 4 \end{bmatrix}$$

Skew-symmetric Matrix:

A square matrix is called skew-symmetric, if it is equal to the opposite of its transpose. For example:

$$\begin{bmatrix} 0 & 8 & -7 \\ -8 & 0 & -0.2 \\ 7 & 0.2 & 0 \end{bmatrix}$$

Note that the diagonal elements of a skew-symmetric matrix should be zero.

Sum of Matrices

Sum of Matrices:

The sum of two matrices of the same order is obtained by adding the corresponding elements.

$$\begin{bmatrix} 1 & 4 \\ 7 & 7 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 11 & 15 \end{bmatrix}$$

Scalar Multiplication:

Multiplying a matrix by a real number is equivalent to multiplying each element of the matrix with the same number.

$$3\begin{bmatrix} 1 & 4 \\ 7 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 21 & 21 \end{bmatrix}$$

Multiplication of Matrices:

The Multiplication of matrix [A] by matrix [B] is possible only if the number of columns of [A] is equal to the number of rows of [B]. The resulting matrix [C] is computed as follows:

$$c_{i,j} = \sum_{k} (a_{i,k} \times b_{k,j})$$

where (i) is the number of rows of [A], (k) is the number of columns of [A], (j) is the number of columns of [B], .

Multiplication of Matrices

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} (1 \times 5 + 2 \times 7) & (1 \times 6 + 2 \times 8) \\ (3 \times 5 + 4 \times 7) & (3 \times 6 + 4 \times 8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 0 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = [(1 \times 3 + 0 \times 2 + 7 \times 1)] = 10$$

Multiplication of Matrices

Example:

$$\begin{bmatrix} 1 & 0 & 7 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= [(1 \times 3 + 0 \times 2 + 7 \times 1) \quad (1 \times 1 + 0 \times 0 + 7 \times 0)] = [10 \quad 1]$$

Example:

$$\begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 7 \end{bmatrix} = ?$$

Determinant

Determinant of a Matrix:

Determinant of a 1x1 matrix:

The determinant of a 1x1 matrix is the element itself.

Determinant of a 2x2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Cofactor

Cofactor of an element in Matrix:

The cofactor COF of an element $a_{i,j}$ is given by:

$$COF\left(a_{i,j}\right) = (-1)^{i+j}|M|$$

where $|M|$ is the determinant of the sub-matrix (Minor Matrix) of [A], obtained from [A] by crossing out the *i*-th row and the *j*-th column.

Cofactor

Example:

$$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\left[COF(a_{1,1})\right] = (-1)^{1+1}|4| = 4$$
$$\left[COF(a_{1,2})\right] = (-1)^{1+2}|3| = -3$$
$$\left[COF(a_{2,1})\right] = (-1)^{2+1}|2| = -2$$
$$\left[COF(a_{2,2})\right] = (-1)^{2+2}|1| = 1$$

$$[CoF(A)] = \begin{bmatrix} (-1)^2|4| & (-1)^3|3| \\ (-1)^3|2| & (-1)^4|1| \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

Adjoint Matrix

Example: $[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$[COF(A)] = \begin{bmatrix} (-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & (-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} & (-1)^4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ (-1)^3 \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} & (-1)^4 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} & (-1)^5 \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} \\ (-1)^4 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & (-1)^5 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} & (-1)^6 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix}$$

$$[COF(A)]$$
$$= \begin{bmatrix} +1 \times (5 \times 9 - 6 \times 8) & -1 \times (4 \times 9 - 6 \times 7) & +1 \times (4 \times 8 - 5 \times 7) \\ -1 \times (2 \times 9 - 3 \times 8) & +1 \times (1 \times 9 - 3 \times 7) & -1 \times (1 \times 8 - 2 \times 7) \\ +1 \times (2 \times 6 - 3 \times 5) & -1 \times (1 \times 6 - 3 \times 4) & +1 \times (1 \times 5 - 2 \times 4) \end{bmatrix}$$

$$[COF(A)] = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

Determinant of a Matrix:

The determinant of a general square matrix (nxn) is calculated by expanding either along any row *i*, according to:

$$|A| = \sum_{j=1}^{n}\left(a_{i,j} \times COF\left(a_{i,j}\right)\right)$$

or along any column *j*, according to:

$$|A| = \sum_{i=1}^{n}\left(a_{i,j} \times COF\left(a_{i,j}\right)\right)$$

Determinant

Example: $[A] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 1 & 3 \end{bmatrix}$

By expanding along row 1, $|A| = \sum_{j=1}^{3} \left(a_{1,j} \times COF(a_{1,j}) \right)$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 1 & 3 \end{vmatrix}$$

$$= 1(-1)^2 \begin{vmatrix} 4 & 5 \\ 1 & 3 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 0 & 5 \\ 7 & 3 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 0 & 4 \\ 7 & 1 \end{vmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 1 & 3 \end{vmatrix} = 1(1)(7) + 2(-1)(-35) + 3(1)(-28) = -7$$

Determinant

Example: $[A] = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 1 & 3 \end{bmatrix}$

By expanding along column 2, $|A| = \sum_{i=1}^{3} \left(a_{i,2} \times COF\left(a_{i,2}\right)\right)$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 1 & 3 \end{vmatrix}$$

$$= 2(-1)^3 \begin{vmatrix} 0 & 5 \\ 7 & 3 \end{vmatrix} + 4(-1)^4 \begin{vmatrix} 1 & 3 \\ 7 & 3 \end{vmatrix} + 1(-1)^5 \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 1 & 3 \end{vmatrix} = 2(-1)(-35) + 4(1)(-18) + 1(-1)(5) = -7$$

Determinant

Properties of determinants:

- A matrix [A] and its transpose have the same determinant.
- If a row of a matrix only consists of zeros, its determinant is 0.
- If a column of a matrix only consists of zeros, its determinant is 0.
- If a matrix has two equal rows or two equal columns, its determinant is 0.
- If a matrix has two proportional rows or two proportional columns, the determinant is 0.
- The determinant of a diagonal matrix is the product of the diagonal elements.
- $|[A]\ [B]| = |A| . |B|$

Adjoint Matrix

Adjoint matrix of a square matrix [A]:

The adjoint matrix of a square matrix [A], is calculated by taking the transpose of the cofactor matrix of [A].

$$[Adj(A)] = [COF(A)]^{\mathrm{T}}$$

Adjoint Matrix

Example: Find the adjoint matrix of

$$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$[COF(A)] = \begin{bmatrix} (-1)^2|4| & (-1)^3|3| \\ (-1)^3|2| & (-1)^4|1| \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$[Adj(A)] = [COF(A)]^{\mathrm{T}} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$[Adj(A)] = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Adjoint Matrix

Example: Find the adjoint matrix of $[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$[COF(A)] = \begin{bmatrix} (-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & (-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} & (-1)^4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ (-1)^3 \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} & (-1)^4 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} & (-1)^5 \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} \\ (-1)^4 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & (-1)^5 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} & (-1)^6 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix}$$

$$[COF(A)] = \begin{bmatrix} +1 \times (5 \times 9 - 6 \times 8) & -1 \times (4 \times 9 - 6 \times 7) & +1 \times (4 \times 8 - 5 \times 7) \\ -1 \times (2 \times 9 - 3 \times 8) & +1 \times (1 \times 9 - 3 \times 7) & -1 \times (1 \times 8 - 2 \times 7) \\ +1 \times (2 \times 6 - 3 \times 5) & -1 \times (1 \times 6 - 3 \times 4) & +1 \times (1 \times 5 - 2 \times 4) \end{bmatrix}$$

$$[COF(A)] = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$$[Adj(A)] = [COF(A)]^{\mathrm{T}} = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

Inverse of a Matrix:

The matrix $[A]^{-1}$ is called the inverse of the <u>square</u> matrix $[A]$, if:

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

where $[I]$ is the identity matrix.

It is computed using:

$$[A]^{-1} = \frac{[Adj(A)]}{|A|}$$

Inverse of a Matrix

Example: Find the inverse matrix of $[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$[COF(A)] = \begin{bmatrix} (-1)^2|4| & (-1)^3|3| \\ (-1)^3|2| & (-1)^4|1| \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$[Adj(A)] = [COF(A)]^{\mathrm{T}} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

$$[A]^{-1} = \frac{[Adj(A)]}{|A|} = \frac{\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}}{-2}$$

$$[A]^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

Inverse of a Matrix

Example: Find the inverse matrix of $[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$[COF(A)] = \begin{bmatrix} (-1)^2|4| & (-1)^3|3| \\ (-1)^3|2| & (-1)^4|1| \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$[Adj(A)] = [CoF(A)]^{\mathrm{T}} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

$$[A]^{-1} = \frac{[Adj(A)]}{|A|} = \frac{\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}}{-2}$$

$$[A]^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

Note:

You can double check by:

$$[A][A]^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

$$[A][A]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[A]^{-1}\,[A] = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$[A]^{-1}\,[A] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse of a Matrix

Example: Find the inverse matrix of $[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$[Adj(A)] = [COF(A)]^{\mathrm{T}} = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 7(-1)^4 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + 8(-1)^5 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + 9(-1)^6 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 7(1)(-3) + 8(-1)(-6) + 9(1)(-3) = 0$$

$$[A]^{-1} = \frac{[Adj(A)]}{|A|} = \frac{\begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}}{0}$$

!!!

A Singular Matrix

A singular matrix:

It is a square matrix for which the determinant is equal to zero.

A singular matrix has no inverse.

Orthogonal Matrix:

This is a square matrix that has its inverse equal to its transpose:

$$[A]^{-1} = [A]^{T}$$

Notes:

- The matrix product of two orthogonal matrices is another orthogonal matrix.
- The inverse of an orthogonal matrix is an orthogonal matrix.

Proper Orthogonal Matrix:

This is an orthogonal matrix that has its determinant equal 1:

$$|A| = 1$$

Rotation Matrix

Rotation Matrix:

This is an orthogonal matrix that is used for rotating vectors or mapping from one coordinate system to another.

Differentiation and Integration

Differentiating a matrix:

A matrix is differentiated by differentiating every element in the matrix.

Integrating a matrix:

A matrix is integrated by integrating every element in the matrix.